UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 2 June 2003 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **4F, 8F** *should be in one bundle and* **1J, 11J** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) Let $(X_n, Y_n)_{n \geqslant 0}$ be a simple symmetric random walk in $\mathbb{Z}^2$, starting from $(0,0)$, and set $T = \inf\{n \geqslant 0 : \max\{|X_n|, |Y_n|\} = 2\}$. Determine the quantities $\mathbb{E}(T)$ and $\mathbb{P}(X_T = 2 \text{ and } Y_T = 0)$.

(ii) Let $(X_n)_{n \geqslant 0}$ be a discrete-time Markov chain with state-space I and transition matrix P. What does it mean to say that a state $i \in I$ is recurrent? Prove that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, where $p_{ii}^{(n)}$ denotes the (i,i) entry in P^n.

Show that the simple symmetric random walk in $\mathbb{Z}^2$ is recurrent.

2D Principles of Dynamics

(i) Consider N particles moving in 3 dimensions. The Cartesian coordinates of these particles are $x^A(t)$, $A = 1, \ldots, 3N$. Now consider an invertible change of coordinates to coordinates $q^a(x^A, t)$, $a = 1, \ldots, 3N$, so that one may express x^A as $x^A(q^a, t)$. Show that the velocity of the system in Cartesian coordinates $\dot{x}^A(t)$ is given by the following expression:

$$\dot{x}^A(\dot{q}^a, q^a, t) = \sum_{b=1}^{3N} \dot{q}^b \frac{\partial x^A}{\partial q^b}(q^a, t) + \frac{\partial x^A}{\partial t}(q^a, t) \,.$$

Furthermore, show that Lagrange's equations in the two coordinate systems are related via

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \sum_{A=1}^{3N} \frac{\partial x^A}{\partial q^a}\left(\frac{\partial L}{\partial x^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A}\right) .$$

(ii) Now consider the case where there are $p < 3N$ constraints applied, $f^\ell(x^A, t) = 0$, $\ell = 1, \ldots, p$. By considering the f^ℓ, $\ell = 1, \ldots, p$, and a set of independent coordinates $q^a, a = 1, \ldots, 3N - p$, as a set of $3N$ new coordinates, show that the Lagrange equations of the constrained system, i.e.

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) + \sum_{\ell=1}^{p} \lambda^\ell \frac{\partial f^\ell}{\partial x^A} = 0, \quad A = 1, \ldots, 3N,$$
$$f^\ell = 0, \qquad \ell = 1, \ldots, p,$$

(where the λ^ℓ are Lagrange multipliers) imply Lagrange's equations for the unconstrained coordinates, i.e.

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0, \quad a = 1, \ldots, 3N - p \,.$$

3G Functional Analysis

(i) Let $T : H_1 \to H_2$ be a continuous linear map between two Hilbert spaces H_1, H_2. Define the adjoint T^* of T. Explain what it means to say that T is Hermitian or unitary.

Let $\phi : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. Show that the map

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

with $Tf(x) = \phi(x)f(x+1)$ is a continuous linear map and find its adjoint. When is T Hermitian? When is it unitary?

(ii) Let C be a closed, non-empty, convex subset of a real Hilbert space H. Show that there exists a unique point $x_o \in C$ with minimal norm. Show that x_o is characterised by the property

$$\langle x_o - x, x_o \rangle \leqslant 0 \qquad \text{for all} \quad x \in C \, .$$

Does this result still hold when C is not closed or when C is not convex? Justify your answers.

4F Groups, Rings and Fields

(i) Let p be a prime number. Show that a group G of order p^n $(n \geqslant 2)$ has a nontrivial normal subgroup, that is, G is not a simple group.

(ii) Let p and q be primes, $p > q$. Show that a group G of order pq has a normal Sylow p-subgroup. If G has also a normal Sylow q-subgroup, show that G is cyclic. Give a necessary and sufficient condition on p and q for the existence of a non-abelian group of order pq. Justify your answer.

5A Electromagnetism

(i) Using Maxwell's equations as they apply to magnetostatics, show that the magnetic field $\mathbf{B}$ can be described in terms of a vector potential $\mathbf{A}$ on which the condition $\nabla \cdot \mathbf{A} = 0$ may be imposed. Hence derive an expression, valid at any point in space, for the vector potential due to a steady current distribution of density $\mathbf{j}$ that is non-zero only within a finite domain.

(ii) Verify that the vector potential $\mathbf{A}$ that you found in Part (i) satisfies $\nabla \cdot \mathbf{A} = 0$, and use it to obtain the Biot–Savart law expression for $\mathbf{B}$. What is the corresponding result for a steady surface current distribution of density $\mathbf{s}$?

In cylindrical polar coordinates (ρ, ϕ, z) (oriented so that $\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_z$) a surface current

$$\mathbf{s} = s(\rho)\mathbf{e}_\phi$$

flows in the plane $z = 0$. Given that

$$s(\rho) = \begin{cases} 4I\left(1 + \frac{a^2}{\rho^2}\right)^{\frac{1}{2}} & a \leqslant \rho \leqslant 3a \\ 0 & \text{otherwise} \end{cases}$$

show that the magnetic field at the point $\mathbf{r} = a\mathbf{e}_z$ has z-component

$$B_z = \mu_0 I \log 5\,.$$

State, with justification, the full result for $\mathbf{B}$ at the point $\mathbf{r} = a\mathbf{e}_z$.

6D Dynamics of Differential Equations

(i) State and prove *Dulac's Criterion* for the non-existence of periodic orbits in $\mathbb{R}^2$. Hence show (choosing a weighting factor of the form $x^\alpha y^\beta$) that there are no periodic orbits of the equations

$$\dot{x} = x(2 - 6x^2 - 5y^2)\ , \quad \dot{y} = y(-3 + 10x^2 + 3y^2)\,.$$

(ii) State the *Poincaré–Bendixson Theorem.* A model of a chemical reaction (the Brusselator) is defined by the second order system

$$\dot{x} = a - x(1 + b) + x^2 y\ , \quad \dot{y} = bx - x^2 y,$$

where a, b are positive parameters. Show that there is a unique fixed point. Show that, for a suitable choice of $p > 0$, trajectories enter the closed region bounded by $x = p$, $y = b/p$, $x + y = a + b/p$ and $y = 0$. Deduce that when $b > 1 + a^2$, the system has a periodic orbit.

7H Logic, Computation and Set Theory

(i) State Zorn's Lemma. Use Zorn's Lemma to prove that every real vector space has a basis.

(ii) State the Bourbaki–Witt Theorem, and use it to prove Zorn's Lemma, making clear where in the argument you appeal to the Axiom of Choice.

Conversely, deduce the Bourbaki–Witt Theorem from Zorn's Lemma.

If X is a non-empty poset in which every chain has an upper bound, must X be chain-complete?

8F Graph Theory

(i) State Brooks' Theorem, and prove it in the case of a 3-connected graph.

(ii) Let G be a bipartite graph, with vertex classes X and Y, each of order n. If G contains no cycle of length 4 show that

$$e(G) \leqslant \frac{n}{2}(1+\sqrt{4n-3}).$$

For which integers $n \leq 12$ are there examples where equality holds?

9G Number Theory

(i) Let p be an odd prime and k a strictly positive integer. Prove that the multiplicative group of relatively prime residue classes modulo p^k is cyclic.

[*You may assume that the result is true for* $k=1$.]

(ii) Let $n = p_1 p_2 \ldots p_r$, where $r \geqslant 2$ and $p_1, p_2, \ldots, p_r$ are distinct odd primes. Let B denote the set of all integers which are relatively prime to n. We recall that n is said to be an *Euler pseudo-prime to the base* $b \in B$ if

$$b^{(n-1)/2} \equiv \left(\frac{b}{n}\right) \mod n\,.$$

If n is an Euler pseudo-prime to the base $b_1 \in B$, but is not an Euler pseudo-prime to the base $b_2 \in B$, prove that n is not an Euler pseudo-prime to the base $b_1 b_2$. Let p denote any of the primes $p_1, p_2, \ldots, p_r$. Prove that there exists a $b \in B$ such that

$$\left(\frac{b}{p}\right) = -1 \qquad \text{and} \qquad b \equiv 1 \mod n/p\,,$$

and deduce that n is not an Euler pseudo-prime to this base b. Hence prove that n is not an Euler pseudo-prime to the base b for at least half of all the relatively prime residue classes b mod n.

10F Coding and Cryptography

(i) We work over the field of two elements. Define what is meant by a linear code of length n. What is meant by a generator matrix for a linear code?

Define what is meant by a parity check code of length n. Show that a code is linear if and only if it is a parity check code.

Give the original Hamming code in terms of parity checks and then find a generator matrix for it.

[*You may use results from the theory of vector spaces provided that you quote them correctly.*]

(ii) Suppose that $1/4 > \delta > 0$ and let $\alpha(n, n\delta)$ be the largest information rate of any binary error correcting code of length n which can correct $[n\delta]$ errors.

Show that

$$1 - H(2\delta) \leqslant \liminf_{n\to\infty} \alpha(n, n\delta) \leqslant 1 - H(\delta)$$

where

$$H(\eta) = -\eta \log_2 \eta - (1-\eta)\log_2(1-\eta).$$

[*You may assume any form of Stirling's theorem provided that you quote it correctly.*]

11J Stochastic Financial Models

(i) In the context of a single-period financial market with d traded assets, what is an *arbitrage?* What is an *equivalent martingale measure?*

A simple single-period financial market contains two assets, S^0 (a bond), and S^1 (a share). The period can be good, bad, or indifferent, with probabilities 1/3 each. At the beginning of the period, time 0, both assets are worth 1, i.e.

$$S_0^0 = 1 = S_0^1,$$

and at the end of the period, time 1, the share is worth

$$S_1^1 = \begin{cases} a & \text{if the period was bad,} \\ b & \text{if the period was indifferent,} \\ c & \text{if the period was good,} \end{cases}$$

where $a < b < c$. The bond is always worth 1 at the end of the period. Show that there is no arbitrage in this market if and only if $a < 1 < c$.

(ii) An agent with C^2 strictly increasing strictly concave utility U has wealth w_0 at time 0, and wishes to invest his wealth in shares and bonds so as to maximise his expected utility of wealth at time 1. Explain how the solution to his optimisation problem generates an equivalent martingale measure.

Assume now that $a = 3/4$, $b = 1$, and $c = 3/2$. Characterise all equivalent martingale measures for this problem. Characterise all equivalent martingale measures which arise as solutions of an agent's optimisation problem.

Calculate the largest and smallest possible prices for a European call option with strike 1 and expiry 1, as the pricing measure ranges over all equivalent martingale measures. Calculate the corresponding bounds when the pricing measure is restricted to the set arising from expected-utility-maximising agents' optimisation problems.

12I Principles of Statistics

(i) A public health official is seeking a rational policy of vaccination against a relatively mild ailment which causes absence from work. Surveys suggest that 60% of the population are already immune, but accurate tests to detect vulnerability in any individual are too costly for mass screening. A simple skin test has been developed, but is not completely reliable. A person who is immune to the ailment will have a negligible reaction to the skin test with probability 0.4, a moderate reaction with probability 0.5 and a strong reaction with probability 0.1. For a person who is vulnerable to the ailment the corresponding probabilities are 0.1, 0.4 and 0.5. It is estimated that the money-equivalent of work-hours lost from failing to vaccinate a vulnerable person is 20, that the unnecessary cost of vaccinating an immune person is 8, and that there is no cost associated with vaccinating a vulnerable person or failing to vaccinate an immune person. On the basis of the skin test, it must be decided whether to vaccinate or not. What is the Bayes decision rule that the health official should adopt?

(ii) A collection of I students each sit J exams. The ability of the ith student is represented by θ_i and the performance of the ith student on the jth exam is measured by X_{ij}. Assume that, given $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_I)$, an appropriate model is that the variables $\{X_{ij}\,,\, 1 \leqslant i \leqslant I,\, 1 \leqslant j \leqslant J\}$ are independent, and

$$X_{ij} \sim N(\theta_i, \tau^{-1}),$$

for a known positive constant τ. It is reasonable to assume, *a priori*, that the θ_i are independent with

$$\theta_i \sim N(\mu, \zeta^{-1}),$$

where μ and ζ are population parameters, known from experience with previous cohorts of students.

Compute the posterior distribution of $\boldsymbol{\theta}$ given the observed exam marks vector $\mathbf{X} = \{X_{ij}\,,\, 1 \leqslant i \leqslant I,\, 1 \leqslant j \leqslant J\}$.

Suppose now that τ is also unknown, but assumed to have a $\text{Gamma}(\alpha_0, \beta_0)$ distribution, for known α_0, β_0. Compute the posterior distribution of τ given $\boldsymbol{\theta}$ and $\mathbf{X}$. Find, up to a normalisation constant, the form of the marginal density of $\boldsymbol{\theta}$ given $\mathbf{X}$.

13I Computational Statistics and Statistical Modelling

(i) Suppose Y_i, $1 \leqslant i \leqslant n$, are independent binomial observations, with $Y_i \sim Bi(t_i, \pi_i)$, $1 \leqslant i \leqslant n$, where $t_1, \ldots, t_n$ are known, and we wish to fit the model

$$\omega : \log \frac{\pi_i}{1 - \pi_i} = \mu + \beta^T x_i \quad \text{for each } i,$$

where $x_1, \ldots, x_n$ are given covariates, each of dimension p. Let $\hat{\mu}$, $\hat{\beta}$ be the maximum likelihood estimators of μ, β. Derive equations for $\hat{\mu}$, $\hat{\beta}$ and state without proof the form of the approximate distribution of $\hat{\beta}$.

(ii) In 1975, data were collected on the 3-year survival status of patients suffering from a type of cancer, yielding the following table

		survive?	
age in years	malignant	yes	no
under 50	no	77	10
under 50	yes	51	13
50-69	no	51	11
50-69	yes	38	20
70+	no	7	3
70+	yes	6	3

Here the second column represents whether the initial tumour was not malignant or was malignant.

Let Y_{ij} be the number surviving, for age group i and malignancy status j, for $i = 1, 2, 3$ and $j = 1, 2$, and let t_{ij} be the corresponding total number. Thus $Y_{11} = 77$, $t_{11} = 87$. Assume $Y_{ij} \sim Bi(t_{ij}, \pi_{ij})$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 2$. The results from fitting the model

$$\log(\pi_{ij}/(1 - \pi_{ij})) = \mu + \alpha_i + \beta_j$$

with $\alpha_1 = 0$, $\beta_1 = 0$ give $\hat{\beta}_2 = -0.7328$ (se $= 0.2985$), and deviance $= 0.4941$. What do you conclude?

Why do we take $\alpha_1 = 0$, $\beta_1 = 0$ in the model?

What "residuals" should you compute, and to which distribution would you refer them?

14C Quantum Physics

(i) An electron of mass m and spin $\frac{1}{2}$ moves freely inside a cubical box of side L . Verify that the energy eigenstates of the system are $\phi_{lmn}(\mathbf{r})\chi_\pm$ where the spatial wavefunction is given by

$$\phi_{lmn}(\mathbf{r}) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{l\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \quad ,$$

and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Give the corresponding energy eigenvalues.

A second electron is inserted into the box. Explain how the Pauli principle determines the structure of the wavefunctions associated with the lowest energy level and the first excited energy level. What are the values of the energy in these two levels and what are the corresponding degeneracies?

(ii) When the side of the box, L, is large, the number of eigenstates available to the electron with energy in the range $E \to E + dE$ is $\rho(E)dE$. Show that

$$\rho(E) = \frac{L^3}{\pi^2\hbar^3}\sqrt{2m^3E} \quad .$$

A large number, N, of electrons are inserted into the box. Explain how the ground state is constructed and define the Fermi energy, E_F. Show that in the ground state

$$N = \frac{2}{3}\frac{L^3}{\pi^2\hbar^3}\sqrt{2m^3}\,(E_F)^{3/2} \quad .$$

When a magnetic field H in the z-direction is applied to the system, an electron with spin up acquires an additional energy $+\mu H$ and an electron with spin down an energy $-\mu H$, where $-\mu$ is the magnetic moment of the electron and $\mu > 0$. Describe, for the case $E_F > \mu H$, the structure of the ground state of the system of N electrons in the box and show that

$$N = \frac{1}{3}\frac{L^3}{\pi^2\hbar^3}\sqrt{2m^3}\left((E_F + \mu H)^{3/2} + (E_F - \mu H)^{3/2}\right) \quad .$$

Calculate the induced magnetic moment, M, of the ground state of the system and show that for a *weak* magnetic field the magnetic moment is given by

$$M \approx \frac{3}{2}N\frac{\mu^2 H}{E_F} \quad .$$

15A General Relativity

(i) The worldline $x^a(\lambda)$ of a massive particle moving in a spacetime with metric g_{ab} obeys the geodesic equation

$$\frac{d^2x^a}{d\tau^2} + \left\{ {}_b{}^a{}_c \right\} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

where τ is the particle's proper time and $\left\{ {}_b{}^a{}_c \right\}$ are the Christoffel symbols; these are the equations of motion for the Lagrangian

$$L_1 = -m\sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$$

where m is the particle's mass, and $\dot{x}^a = dx^a/d\lambda$. Why is the choice of worldline parameter λ irrelevant? Among all possible worldlines passing through points A and B, why is $x^a(\lambda)$ the one that extremizes the proper time elapsed between A and B?

Explain how the equations of motion for a massive particle may be obtained from the alternative Lagrangian

$$L_2 = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b\,.$$

What can you conclude from the fact that L_2 has no explicit dependence on λ? How are the equations of motion for a massless particle obtained from L_2?

(ii) A photon moves in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(1 - \frac{2M}{r}\right) dt^2.$$

Given that the motion is confined to the plane $\theta = \pi/2$, obtain the radial equation

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right),$$

where E and h are constants, the physical meaning of which should be stated.

Setting $u = 1/r$, obtain the equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2.$$

Using the approximate solution

$$u = \frac{1}{b}\sin\phi + \frac{M}{2b^2}\left(3 + \cos 2\phi\right) + \ldots,$$

obtain the standard formula for the deflection of light passing far from a body of mass M with impact parameter b. Reinstate factors of G and c to give your result in physical units.

16A Statistical Physics and Cosmology

(i) Explain briefly how the relative motion of galaxies in a homogeneous and isotropic universe is described in terms of the scale factor $a(t)$ (where t is time). In particular, show that the relative velocity $\mathbf{v}(t)$ of two galaxies is given in terms of their relative displacement $\mathbf{r}(t)$ by the formula $\mathbf{v}(t) = H(t)\mathbf{r}(t)$, where $H(t)$ is a function that you should determine in terms of $a(t)$. Given that $a(0) = 0$, obtain a formula for the distance $R(t)$ to the cosmological horizon at time t. Given further that $a(t) = (t/t_0)^{\alpha}$, for $0 < \alpha < 1$ and constant t_0, compute $R(t)$. Hence show that $R(t)/a(t) \to 0$ as $t \to 0$.

(ii) A homogeneous and isotropic model universe has energy density $\rho(t)c^2$ and pressure $P(t)$, where c is the speed of light. The evolution of its scale factor $a(t)$ is governed by the Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$

where the overdot indicates differentiation with respect to t. Use the "Fluid" equation

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\left(\frac{\dot{a}}{a}\right)$$

to obtain an equation for the acceleration $\ddot{a}(t)$. Assuming $\rho > 0$ and $P \geqslant 0$, show that ρa^3 cannot increase with time as long as $\dot{a} > 0$, nor decrease if $\dot{a} < 0$. Hence determine the late time behaviour of $a(t)$ for $k < 0$. For $k > 0$ show that an initially expanding universe must collapse to a "big crunch" at which $a \to 0$. How does $\dot{a}$ behave as $a \to 0$? Given that $P = 0$, determine the form of $a(t)$ near the big crunch. Discuss the qualitative late time behaviour for $k = 0$.

Cosmological models are often assumed to have an equation of state of the form $P = \sigma\rho c^2$ for constant σ. What physical principle requires $\sigma \leqslant 1$? Matter with $P = \rho c^2$ ($\sigma = 1$) is called "stiff matter" by cosmologists. Given that $k = 0$, determine $a(t)$ for a universe that contains only stiff matter. In our Universe, why would you expect stiff matter to be negligible now even if it were significant in the early Universe?

17C Symmetries and Groups in Physics

(i) Define the character χ of a representation D of a finite group G. Show that $<\chi \mid \chi>= 1$ if and only if D is irreducible, where

$$<\chi \mid \chi>= \frac{1}{\mid G \mid}\sum_{g \in G} \chi(g)\chi(g^{-1})\,.$$

If $\mid G \mid= 8$ and $<\chi \mid \chi>= 2$, what are the possible dimensions of the representation D?

(ii) State and prove Schur's first and second lemmas.

18E Transport Processes

(i) A solute occupying a domain V_0 has concentration $C(\boldsymbol{x},t)$ and is created at a rate $S(\boldsymbol{x},t)$ per unit volume; $\boldsymbol{J}(\boldsymbol{x},t)$ is the flux of solute per unit area; $\boldsymbol{x},t$ are position and time. Derive the transport equation

$$C_t + \nabla\cdot\boldsymbol{J} = S.$$

State Fick's Law of diffusion and hence write down the diffusion equation for $C(\boldsymbol{x},t)$ for a case in which the solute flux occurs solely by diffusion, with diffusivity $D(\boldsymbol{x})$.

In a finite domain $0 \leqslant x \leqslant L$, D, S and the steady-state distribution of C depend only on x; C is equal to C_0 at $x=0$ and $C_1 \neq C_0$ at $x=L$. Find $C(x)$ in the following two cases:

(a) $D = D_0$, $S = 0$,

(b) $D = D_1 x^{1/2}$, $S = 0$,

where D_0 and D_1 are positive constants.

Show that there is no steady solution satisfying the boundary conditions if $D = D_1 x$, $S = 0$.

(ii) For the problem of Part (i), consider the case $D = D_0$, $S = kC$, where D_0 and k are positive constants. Calculate the steady-state solution, $C = C_s(x)$, assuming that $\sqrt{k/D_0} \neq n\pi/L$ for any integer n.

Now let

$$C(x,0) = C_0 \frac{\sin\alpha(L-x)}{\sin\alpha L},$$

where $\alpha = \sqrt{k/D_0}$. Find the equations, boundary and initial conditions satisfied by $C'(x,t) = C(x,t) - C_s(x)$. Solve the problem using separation of variables and show that

$$C'(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \exp\left[\left(\alpha^2 - \frac{n^2\pi^2}{L^2}\right) D_0 t\right],$$

for some constants A_n. Write down an integral expression for A_n, show that

$$A_1 = -\frac{2\pi C_1}{\alpha^2 L^2 - \pi^2},$$

and comment on the behaviour of the solution for large times in the two cases $\alpha L < \pi$ and $\alpha L > \pi$.

19E Theoretical Geophysics

(i) Explain the concepts of: traction on an element of surface; the stress tensor; the strain tensor in an elastic medium. Derive a relationship between the two tensors for a linear isotropic elastic medium, stating clearly any assumption you need to make.

(ii) State what is meant by an SH wave in a homogeneous isotropic elastic medium. An SH wave in a medium with shear modulus μ and density ρ is incident at angle θ on an interface with a medium with shear modulus μ' and density ρ'. Evaluate the form and amplitude of the reflected wave and transmitted wave. Comment on the case $c' \sin\theta / c > 1$, where $c^2 = \mu/\rho$ and $(c')^2 = \mu'/\rho'$.

20E Numerical Analysis

(i) The linear algebraic equations $A\mathbf{u} = \mathbf{b}$, where A is symmetric and positive-definite, are solved with the Gauss–Seidel method. Prove that the iteration always converges.

(ii) The Poisson equation $\nabla^2 u = f$ is given in the bounded, simply connected domain $\Omega \subseteq \mathbb{R}^2$, with zero Dirichlet boundary conditions on $\partial\Omega$. It is approximated by the five-point formula

$$U_{m-1,n} + U_{m,n-1} + U_{m+1,n} + U_{m,n+1} - 4U_{m,n} = (\Delta x)^2 f_{m,n},$$

where $U_{m,n} \approx u(m\Delta x, n\Delta x)$, $\quad f_{m,n} = f(m\Delta x, n\Delta x)$, and $(m\Delta x, n\Delta x)$ is in the interior of Ω.

Assume for the sake of simplicity that the intersection of $\partial\Omega$ with the grid consists only of grid points, so that no special arrangements are required near the boundary. Prove that the method can be written in a vector notation, $A\mathbf{u} = \mathbf{b}$ with a negative-definite matrix A.