UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 3 June 2003 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** ..., **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Let E be a subset of $\mathbb{R}^n$. Prove that the following conditions on E are equivalent:

(i) E is closed and bounded.

(ii) E has the Bolzano–Weierstrass property (i.e., every sequence in E has a subsequence convergent to a point of E).

(iii) Every continuous real-valued function on E is bounded.

[*The Bolzano–Weierstrass property for bounded closed intervals in* $\mathbb{R}^1$ *may be assumed.*]

2D Methods

Fermat's principle of optics states that the path of a light ray connecting two points will be such that the travel time t is a minimum. If the speed of light varies continuously in a medium and is a function $c(y)$ of the distance from the boundary $y = 0$, show that the path of a light ray is given by the solution to

$$c(y)y'' + c'(y)(1 + y'^2) = 0\,,$$

where $y' = \frac{dy}{dx}$, etc. Show that the path of a light ray in a medium where the speed of light c is a constant is a straight line. Also find the path from $(0, 0)$ to $(1, 0)$ if $c(y) = y$, and sketch it.

3H Statistics

Derive the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ for the coefficients of the simple linear regression model

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are given constants, $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$, and ε_i are independent with $\mathrm{E}\,\varepsilon_i = 0$, $\mathrm{Var}\,\varepsilon_i = \sigma^2$, $i = 1, \ldots, n$.

A manufacturer of optical equipment has the following data on the unit cost (in pounds) of certain custom-made lenses and the number of units made in each order:

No. of units, x_i	1	3	5	10	12
Cost per unit, y_i	58	55	40	37	22

Assuming that the conditions underlying simple linear regression analysis are met, estimate the regression coefficients and use the estimated regression equation to predict the unit cost in an order for 8 of these lenses.

[*Hint: for the data above,* $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})y_i = -257.4$.]

4F Geometry

Describe the geodesics (that is, hyperbolic straight lines) in **either** the disc model **or** the half-plane model of the hyperbolic plane. Explain what is meant by the statements that two hyperbolic lines are parallel, and that they are ultraparallel.

Show that two hyperbolic lines l and l' have a unique common perpendicular if and only if they are ultraparallel.

[*You may assume standard results about the group of isometries of whichever model of the hyperbolic plane you use.*]

5E Linear Mathematics

Let V be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \ .$$

Prove that V is a subspace of $\mathbb{R}^5$. Solve the above equations for a_1 and a_2 in terms of a_3, a_4 and a_5. Hence, exhibit a basis for V, explaining carefully why the vectors you give form a basis.

6C Fluid Dynamics

An unsteady fluid flow has velocity field given in Cartesian coordinates (x, y, z) by $\mathbf{u} = (1, xt, 0)$, where t denotes time. Dye is released into the fluid from the origin continuously. Find the position at time t of the dye particle that was released at time s and hence show that the dye streak lies along the curve

$$y = \tfrac{1}{2}tx^2 - \tfrac{1}{6}x^3.$$

7B Complex Methods

Let $u(x, y)$ and $v(x, y)$ be a pair of conjugate harmonic functions in a domain D. Prove that

$$U(x, y) = e^{-2uv}\cos(u^2 - v^2) \qquad \text{and} \qquad V(x, y) = e^{-2uv}\sin(u^2 - v^2)$$

also form a pair of conjugate harmonic functions in D.

8G Quadratic Mathematics

Let U and V be finite-dimensional vector spaces. Suppose that b and c are bilinear forms on $U \times V$ and that b is non-degenerate. Show that there exist linear endomorphisms S of U and T of V such that $c(x, y) = b(S(x), y) = b(x, T(y))$ for all $(x, y) \in U \times V$.

9A Quantum Mechanics

A particle of mass m is confined inside a one-dimensional box of length a. Determine the possible energy eigenvalues.

SECTION II

10F Analysis II

Explain briefly what is meant by a *metric space*, and by a *Cauchy sequence* in a metric space.

A function $d : X \times X \to \mathbb{R}$ is called a pseudometric on X if it satisfies all the conditions for a metric except the requirement that $d(x, y) = 0$ implies $x = y$. If d is a pseudometric on X, show that the binary relation R on X defined by $x \, R \, y \Leftrightarrow d(x, y) = 0$ is an equivalence relation, and that the function d induces a metric on the set X/R of equivalence classes.

Now let (X, d) be a metric space. If (x_n) and (y_n) are Cauchy sequences in X, show that the sequence whose nth term is $d(x_n, y_n)$ is a Cauchy sequence of real numbers. Deduce that the function $\overline{d}$ defined by

$$\overline{d}((x_n), (y_n)) = \lim_{n \to \infty} d(x_n, y_n)$$

is a pseudometric on the set C of all Cauchy sequences in X. Show also that there is an isometric embedding (that is, a distance-preserving mapping) $X \to C/R$, where R is the equivalence relation on C induced by the pseudometric $\overline{d}$ as in the previous paragraph. Under what conditions on X is $X \to C/R$ bijective? Justify your answer.

11D Methods

(a) Determine the Green's function $G(x, \xi)$ for the operator $\frac{d^2}{dx^2} + k^2$ on $[0, \pi]$ with Dirichlet boundary conditions by solving the boundary value problem

$$\frac{d^2 G}{dx^2} + k^2 G = \delta(x - \xi) \;, \quad G(0) = 0, \; G(\pi) = 0$$

when k is not an integer.

(b) Use the method of Green's functions to solve the boundary value problem

$$\frac{d^2 y}{dx^2} + k^2 y = f(x) \;, \quad y(0) = a, \; y(\pi) = b$$

when k is not an integer.

12H Statistics

Suppose that six observations $X_1, \ldots, X_6$ are selected at random from a normal distribution for which both the mean μ_X and the variance σ_X^2 are unknown, and it is found that $S_{XX} = \sum_{i=1}^{6}(x_i - \bar{x})^2 = 30$, where $\bar{x} = \frac{1}{6}\sum_{i=1}^{6} x_i$. Suppose also that 21 observations $Y_1, \ldots, Y_{21}$ are selected at random from another normal distribution for which both the mean μ_Y and the variance σ_Y^2 are unknown, and it is found that $S_{YY} = 40$. Derive carefully the likelihood ratio test of the hypothesis H_0: $\sigma_X^2 = \sigma_Y^2$ against H_1: $\sigma_X^2 > \sigma_Y^2$ and apply it to the data above at the 0.05 level.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_{20}^2	χ_{21}^2	$F_{5,20}$	$F_{6,21}$
95% percentile	11.07	12.59	31.41	32.68	2.71	2.57

]

13F Geometry

Write down the Riemannian metric in the half-plane model of the hyperbolic plane. Show that Möbius transformations mapping the upper half-plane to itself are isometries of this model.

Calculate the hyperbolic distance from ib to ic, where b and c are positive real numbers. Assuming that the hyperbolic circle with centre ib and radius r is a Euclidean circle, find its Euclidean centre and radius.

Suppose that a and b are positive real numbers for which the points ib and $a + ib$ of the upper half-plane are such that the hyperbolic distance between them coincides with the Euclidean distance. Obtain an expression for b as a function of a. Hence show that, for any b with $0 < b < 1$, there is a unique positive value of a such that the hyperbolic distance between ib and $a + ib$ coincides with the Euclidean distance.

14E Linear Mathematics

(a) Let U, U' be subspaces of a finite-dimensional vector space V. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let V and W be finite-dimensional vector spaces and let α and β be linear maps from V to W. Prove that

$$\operatorname{rank}(\alpha + \beta) \leqslant \operatorname{rank} \alpha + \operatorname{rank} \beta \ .$$

(c) Deduce from this result that

$$\operatorname{rank}(\alpha + \beta) \geqslant |\operatorname{rank} \alpha - \operatorname{rank} \beta| \ .$$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leqslant r \leqslant s \leqslant n$. Exhibit linear maps $\alpha, \beta \colon V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank} \alpha = r$, $\operatorname{rank} \beta = s$ and $\operatorname{rank}(\alpha + \beta) = n$.

15C Fluid Dynamics

Starting from the Euler equations for incompressible, inviscid flow

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

derive the vorticity equation governing the evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Consider the flow

$$\mathbf{u} = \beta(-x, -y, 2z) + \Omega(t)(-y, x, 0),$$

in Cartesian coordinates (x, y, z), where t is time and β is a constant. Compute the vorticity and show that it evolves in time according to

$$\boldsymbol{\omega} = \omega_0 \mathrm{e}^{2\beta t}\mathbf{k},$$

where ω_0 is the initial magnitude of the vorticity and $\mathbf{k}$ is a unit vector in the z-direction.

Show that the material curve $C(t)$ that takes the form

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 1$$

at $t = 0$ is given later by

$$x^2 + y^2 = a^2(t) \quad \text{and} \quad z = \frac{1}{a^2(t)},$$

where the function $a(t)$ is to be determined.

Calculate the circulation of $\mathbf{u}$ around C and state how this illustrates Kelvin's circulation theorem.

16B Complex Methods

Sketch the region A which is the intersection of the discs

$$D_0 = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : |z - (1+i)| < 1\}.$$

Find a conformal mapping that maps A onto the right half-plane $H = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. Also find a conformal mapping that maps A onto D_0.

[*Hint: You may find it useful to consider maps of the form* $w(z) = \frac{az+b}{cz+d}$.]

17G Quadratic Mathematics

(a) Suppose p is an odd prime and a an integer coprime to p. Define the *Legendre symbol* $\left(\frac{a}{p}\right)$ and state Euler's criterion.

(b) Compute $\left(\frac{-1}{p}\right)$ and prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

whenever a and b are coprime to p.

(c) Let n be any integer such that $1 \leqslant n \leqslant p-2$. Let m be the unique integer such that $1 \leqslant m \leqslant p-2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+m}{p}\right).$$

(d) Find

$$\sum_{n=1}^{p-2} \left(\frac{n(n+1)}{p}\right).$$

18A Quantum Mechanics

What is the significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x)\; Q\; \psi(x) dx$$

of an observable Q in the normalized state $\psi(x)$? Let Q and P be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \tfrac{1}{4} |\langle [Q,P] \rangle|^2 .$$

The uncertainty ΔQ of Q in the state $\psi(x)$ is defined as

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle .$$

Deduce the generalized uncertainty relation,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2} |\langle [Q,P] \rangle| .$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x,p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle \geqslant \tfrac{1}{2} \hbar\omega .$$