UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 2 June 2003 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **C** *and* **E** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Numbers and Sets

(i) Prove by induction or otherwise that for every $n \geqslant 1$,

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r \right)^2 .$$

(ii) Show that the sum of the first n positive cubes is divisible by 4 if and only if $n \equiv 0$ or $3 \pmod 4$.

2C Numbers and Sets

What is an *equivalence relation*? For each of the following pairs $(X, \sim)$, determine whether or not $\sim$ is an equivalence relation on X:

(i) $X = \mathbb{R}$, $x \sim y$ iff $x - y$ is an even integer;

(ii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{R}$;

(iii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{Z}$;

(iv) $X = \mathbb{Z} \setminus \{0\}$, $x \sim y$ iff $x^2 - y^2$ is ± 1 times a perfect square.

3E Dynamics

Because of an accident on launching, a rocket of unladen mass M lies horizontally on the ground. It initially contains fuel of mass m_0, which ignites and is emitted horizontally at a constant rate and at uniform speed u relative to the rocket. The rocket is initially at rest. If the coefficient of friction between the rocket and the ground is μ, and the fuel is completely burnt in a total time T, show that the final speed of the rocket is

$$u \log \left(\frac{M + m_0}{M} \right) - \mu g T.$$

4E Dynamics

Write down an expression for the total momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$ with respect to an origin O of a system of n point particles of masses m_i, position vectors (with respect to O) $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, $i = 1, \ldots, n$.

Show that with respect to a new origin O' the total momentum $\mathbf{P}'$ and total angular momentum $\mathbf{L}'$ are given by

$$\mathbf{P}' = \mathbf{P}, \qquad \mathbf{L}' = \mathbf{L} - \mathbf{b} \times \mathbf{P},$$

and hence

$$\mathbf{L}' \cdot \mathbf{P}' = \mathbf{L} \cdot \mathbf{P},$$

where $\mathbf{b}$ is the constant vector displacement of O' with respect to O. How does $\mathbf{L} \times \mathbf{P}$ change under change of origin?

Hence show that **either**

(1) the total momentum vanishes and the total angular momentum is independent of origin, **or**

(2) by choosing $\mathbf{b}$ in a way that should be specified, the total angular momentum with respect to O' can be made parallel to the total momentum.

SECTION II

5C Numbers and Sets

Define what is meant by the term *countable*. Show directly from your definition that if X is countable, then so is any subset of X.

Show that $\mathbb{N} \times \mathbb{N}$ is countable. Hence or otherwise, show that a countable union of countable sets is countable. Show also that for any $n \geqslant 1$, $\mathbb{N}^n$ is countable.

A function $f : \mathbb{Z} \to \mathbb{N}$ is *periodic* if there exists a positive integer m such that, for every $x \in \mathbb{Z}$, $f(x+m) = f(x)$. Show that the set of periodic functions $f : \mathbb{Z} \to \mathbb{N}$ is countable.

6C Numbers and Sets

(i) Prove Wilson's theorem: if p is prime then $(p-1)! \equiv -1 \pmod{p}$.

Deduce that if $p \equiv 1 \pmod{4}$ then

$$\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 \pmod{p}.$$

(ii) Suppose that p is a prime of the form $4k+3$. Show that if $x^4 \equiv 1 \pmod{p}$ then $x^2 \equiv 1 \pmod{p}$.

(iii) Deduce that if p is an odd prime, then the congruence

$$x^2 \equiv -1 \pmod{p}$$

has exactly two solutions (modulo p) if $p \equiv 1 \pmod{4}$, and none otherwise.

7C Numbers and Sets

Let m, n be integers. Explain what is their *greatest common divisor* (m,n). Show from your definition that, for any integer k, $(m,n) = (m+kn, n)$.

State Bezout's theorem, and use it to show that if p is prime and p divides mn, then p divides at least one of m and n.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... is defined by $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$ for $n \geqslant 1$. Prove:

(i) $(x_{n+1}, x_n) = 1$ and $(x_{n+2}, x_n) = 1$ for every $n \geqslant 0$;

(ii) $x_{n+3} \equiv x_n \pmod{2}$ and $x_{n+8} \equiv x_n \pmod{3}$ for every $n \geqslant 0$;

(iii) if $n \equiv 0 \pmod{5}$ then $x_n \equiv 0 \pmod{5}$.

8C Numbers and Sets

Let X be a finite set with n elements. How many functions are there from X to X? How many relations are there on X?

Show that the number of relations R on X such that, for each $y \in X$, there exists at least one $x \in X$ with xRy, is $(2^n - 1)^n$.

Using the inclusion–exclusion principle or otherwise, deduce that the number of such relations R for which, in addition, for each $x \in X$, there exists at least one $y \in X$ with xRy, is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n .$$

9E Dynamics

Write down the equation of motion for a point particle with mass m, charge e, and position vector $\mathbf{x}(t)$ moving in a time-dependent magnetic field $\mathbf{B}(\mathbf{x}, t)$ with vanishing electric field, and show that the kinetic energy of the particle is constant. If the magnetic field is constant in direction, show that the component of velocity in the direction of $\mathbf{B}$ is constant. Show that, in general, the angular momentum of the particle is not conserved.

Suppose that the magnetic field is independent of time and space and takes the form $\mathbf{B} = (0, 0, B)$ and that $\dot{A}$ is the rate of change of area swept out by a radius vector joining the origin to the projection of the particle's path on the (x, y) plane. Obtain the equation

$$\frac{d}{dt}\left(m\dot{A} + \frac{eBr^2}{4} \right) = 0 \,, \qquad (*)$$

where (r, θ) are plane polar coordinates. Hence obtain an equation replacing the equation of conservation of angular momentum.

Show further, using energy conservation and $(*)$, that the equations of motion in plane polar coordinates may be reduced to the first order non-linear system

$$\dot{r} = \sqrt{v^2 - \left(\frac{2c}{mr} - \frac{erB}{2m} \right)^2} ,$$

$$\dot{\theta} = \frac{2c}{mr^2} - \frac{eB}{2m} \,,$$

where v and c are constants.

10E Dynamics

Write down the equations of motion for a system of n gravitating particles with masses m_i, and position vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$.

The particles undergo a motion for which $\mathbf{x}_i(t) = a(t)\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Show that the equations of motion will be satisfied as long as the function $a(t)$ satisfies

$$\ddot{a} = -\frac{\Lambda}{a^2}\ , \qquad (*)$$

where Λ is a constant and the vectors $\mathbf{a}_i$ satisfy

$$\Lambda m_i \mathbf{a}_i = \mathbf{G}_i = \sum_{j \neq i} \frac{G m_i m_j \big(\mathbf{a}_i - \mathbf{a}_j\big)}{|\mathbf{a}_i - \mathbf{a}_j|^3}\ . \qquad (**)$$

Show that $(*)$ has as first integral

$$\frac{\dot{a}^2}{2} - \frac{\Lambda}{a} = \frac{k}{2}\ ,$$

where k is another constant. Show that

$$\mathbf{G}_i = \boldsymbol{\nabla}_i W\ ,$$

where $\boldsymbol{\nabla}_i$ is the gradient operator with respect to $\mathbf{a}_i$ and

$$W = -\sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|}\ .$$

Using Euler's theorem for homogeneous functions (see below), or otherwise, deduce that

$$\sum_i \mathbf{a}_i \cdot \mathbf{G}_i = -W\ .$$

Hence show that all solutions of $(**)$ satisfy

$$\Lambda I = -W$$

where

$$I = \sum_i m_i \mathbf{a}_i^2\ .$$

Deduce that Λ must be positive and that the total kinetic energy plus potential energy of the system of particles is equal to $\dfrac{k}{2}I$.

[*Euler's theorem states that if*

$$f(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^p f(x, y, z, \ldots)\ ,$$

then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \ldots = pf\ .]$$

11E Dynamics

State the parallel axis theorem and use it to calculate the moment of inertia of a uniform hemisphere of mass m and radius a about an axis through its centre of mass and parallel to the base.

[*You may assume that the centre of mass is located at a distance* $\frac{3}{8}a$ *from the flat face of the hemisphere, and that the moment of inertia of a full sphere about its centre is* $\frac{2}{5}Ma^2$, *with* $M = 2m$.]

The hemisphere initially rests on a rough horizontal plane with its base vertical. It is then released from rest and subsequently rolls on the plane without slipping. Let θ be the angle that the base makes with the horizontal at time t. Express the instantaneous speed of the centre of mass in terms of b and the rate of change of θ, where b is the instantaneous distance from the centre of mass to the point of contact with the plane. Hence write down expressions for the kinetic energy and potential energy of the hemisphere and deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{(28 - 15\cos\theta)a} .$$

12E Dynamics

Let (r, θ) be plane polar coordinates and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ unit vectors in the direction of increasing r and θ respectively. Show that the velocity of a particle moving in the plane with polar coordinates $\big(r(t),\ \theta(t)\big)$ is given by

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta ,$$

and that the unit normal $\mathbf{n}$ to the particle path is parallel to

$$r\dot{\theta}\mathbf{e}_r - \dot{r}\mathbf{e}_\theta .$$

Deduce that the perpendicular distance p from the origin to the tangent of the curve $r = r(\theta)$ is given by

$$\frac{r^2}{p^2} = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

The particle, whose mass is m, moves under the influence of a central force with potential $V(r)$. Use the conservation of energy E and angular momentum h to obtain the equation

$$\frac{1}{p^2} = \frac{2m\big(E - V(r)\big)}{h^2} .$$

Hence express θ as a function of r as the integral

$$\theta = \int \frac{hr^{-2}dr}{\sqrt{2m\big(E - V_{\text{eff}}(r)\big)}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{h^2}{2mr^2} .$$

Evaluate the integral and describe the orbit when $V(r) = \dfrac{c}{r^2}$, with c a positive constant.