UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 30 May 2003 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **D** *and* **F** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Differential Equations

Consider the equation

$$\frac{dy}{dx} = 1 - y^2 \,. \qquad (*)$$

Using small line segments, sketch the flow directions in $x \geqslant 0$, $-2 \leqslant y \leqslant 2$ implied by the right-hand side of $(*)$. Find the general solution

(i) in $|y| < 1$,

(ii) in $|y| > 1$.

Sketch a solution curve in each of the three regions $y > 1$, $|y| < 1$, and $y < -1$.

2D Differential Equations

Consider the differential equation

$$\frac{dx}{dt} + Kx = 0 \,,$$

where K is a positive constant. By using the approximate finite-difference formula

$$\frac{dx_n}{dt} = \frac{x_{n+1} - x_{n-1}}{2\delta t} \,,$$

where δt is a positive constant, and where x_n denotes the function $x(t)$ evaluated at $t = n\delta t$ for integer n, convert the differential equation to a difference equation for x_n.

Solve both the differential equation and the difference equation for general initial conditions. Identify those solutions of the difference equation that agree with solutions of the differential equation over a finite interval $0 \leqslant t \leqslant T$ in the limit $\delta t \to 0$, and demonstrate the agreement. Demonstrate that the remaining solutions of the difference equation cannot agree with the solution of the differential equation in the same limit.

[*You may use the fact that, for bounded* $|u|$, $\lim_{\epsilon \to 0} (1 + \epsilon u)^{1/\epsilon} = e^u$.]

3F Probability

(a) Define the *probability generating function* of a random variable. Calculate the probability generating function of a binomial random variable with parameters n and p, and use it to find the mean and variance of the random variable.

(b) X is a binomial random variable with parameters n and p, Y is a binomial random variable with parameters m and p, and X and Y are independent. Find the distribution of $X + Y$; that is, determine $P\{X + Y = k\}$ for all possible values of k.

4F Probability

The random variable X is uniformly distributed on the interval $[0,1]$. Find the distribution function and the probability density function of Y, where

$$Y = \frac{3X}{1-X}\,.$$

SECTION II

5D Differential Equations

(a) Show that if $\mu(x,y)$ is an integrating factor for an equation of the form

$$f(x,y)\,dy \;+\; g(x,y)\,dx \;=\; 0$$

then $\partial(\mu f)/\partial x \;=\; \partial(\mu g)/\partial y\,.$

Consider the equation

$$\cot x\;dy \;-\; \tan y\;dx \;=\; 0$$

in the domain $0 \leqslant x \leqslant \frac{1}{2}\pi\,,\; 0 \leqslant y \leqslant \frac{1}{2}\pi\,.$ Using small line segments, sketch the flow directions in that domain. Show that $\sin x\,\cos y$ is an integrating factor for the equation. Find the general solution of the equation, and sketch the family of solutions that occupies the larger domain $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi\,,\; -\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi\,.$

(b) The following example illustrates that the concept of integrating factor extends to higher-order equations. Multiply the equation

$$\left[y\,\frac{d^2y}{dx^2} \;+\; \left(\frac{dy}{dx}\right)^2\right]\cos^2 x \;=\; 1$$

by $\sec^2 x$, and show that the result takes the form $\dfrac{d}{dx}h(x,y) = 0$, for some function $h(x,y)$ to be determined. Find a particular solution $y = y(x)$ such that $y(0) = 0$ with dy/dx finite at $x = 0$, and sketch its graph in $0 \leqslant x < \frac{1}{2}\pi\,.$

6D Differential Equations

Define the *Wronskian* $W(x)$ associated with solutions of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y \; = \; 0$$

and show that

$$W(x) \;\propto\; \exp\left(-\int^x p(\xi)\,d\xi\right) \; .$$

Evaluate the expression on the right when $p(x) = -2/x$.

Given that $p(x) = -2/x$ and that $q(x) = -1$, show that solutions in the form of power series,

$$y \; = \; x^\lambda \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0)\,,$$

can be found if and only if $\lambda = 0$ or 3. By constructing and solving the appropriate recurrence relations, find the coefficients a_n for each power series.

You may assume that the equation is satisfied by $y = \cosh x - x\sinh x$ and by $y = \sinh x - x\cosh x$. Verify that these two solutions agree with the two power series found previously, and that they give the $W(x)$ found previously, up to multiplicative constants.

[*Hint:* $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$, $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$.]

7D Differential Equations

Consider the linear system

$$\dot{\mathbf{x}}(t) - A\mathbf{x}(t) \;=\; \mathbf{z}(t)$$

where the n-vector $\mathbf{z}(t)$ and the $n \times n$ matrix A are given; A has constant real entries, and has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and n linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$. Find the complementary function. Given a particular integral $\mathbf{x}_{\mathrm{p}}(t)$, write down the general solution. In the case $n = 2$ show that the complementary function is purely oscillatory, with no growth or decay, if and only if

$$\operatorname{trace} A = 0 \qquad \text{and} \qquad \det A > 0 \; .$$

Consider the same case $n = 2$ with $\operatorname{trace} A = 0$ and $\det A > 0$ and with

$$\mathbf{z}(t) = \mathbf{a}_1 \exp(i\omega_1 t) + \mathbf{a}_2 \exp(i\omega_2 t) \, ,$$

where $\omega_1, \; \omega_2$ are given real constants. Find a particular integral when

(i) $i\omega_1 \neq \lambda_1$ and $i\omega_2 \neq \lambda_2$;

(ii) $i\omega_1 \neq \lambda_1$ but $i\omega_2 = \lambda_2$.

In the case

$$A \;=\; \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

with $\mathbf{z}(t) = \begin{pmatrix} 2 \\ 3i - 1 \end{pmatrix} \exp(3it)$, find the solution subject to the initial condition $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

8D Differential Equations

For all solutions of

$$\begin{aligned}\dot{x} &= \tfrac{1}{2}\alpha x + y - 2y^3, \\ \dot{y} &= -x\end{aligned}$$

show that $dK/dt = \alpha x^2$ where

$$K = K(x,y) = x^2 + y^2 - y^4 \ .$$

In the case $\alpha = 0$, analyse the properties of the critical points and sketch the phase portrait, including the special contours for which $K(x,y) = \frac{1}{4}$. Comment on the asymptotic behaviour, as $t \to \infty$, of solution trajectories that pass near each critical point, indicating whether or not any such solution trajectories approach from, or recede to, infinity.

Briefly discuss how the picture changes when α is made small and positive, using your result for dK/dt to describe, in qualitative terms, how solution trajectories cross K-contours.

9F Probability

State the inclusion-exclusion formula for the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs.

After a party the n guests take coats randomly from a pile of their n coats. Calculate the probability that no-one goes home with the correct coat.

Let $p(m,n)$ be the probability that exactly m guests go home with the correct coats. By relating $p(m,n)$ to $p(0,n-m)$, or otherwise, determine $p(m,n)$ and deduce that

$$\lim_{n\to\infty} p(m,n) = \frac{1}{em!} \ .$$

10F Probability

The random variables X and Y each take values in $\{0,1\}$, and their joint distribution $p(x,y) = P\{X = x,\ Y = y\}$ is given by

$$p(0,0) = a, \qquad p(0,1) = b, \qquad p(1,0) = c, \qquad p(1,1) = d\,.$$

Find necessary and sufficient conditions for X and Y to be

(i) uncorrelated;

(ii) independent.

Are the conditions established in (i) and (ii) equivalent?

11F Probability

A laboratory keeps a population of aphids. The probability of an aphid passing a day uneventfully is $q < 1$. Given that a day is not uneventful, there is probability r that the aphid will have one offspring, probability s that it will have two offspring and probability t that it will die, where $r + s + t = 1$. Offspring are ready to reproduce the next day. The fates of different aphids are independent, as are the events of different days. The laboratory starts out with one aphid.

Let X_1 be the number of aphids at the end of the first day. What is the expected value of X_1? Determine an expression for the probability generating function of X_1.

Show that the probability of extinction does not depend on q, and that if $2r+3s \leqslant 1$ then the aphids will certainly die out. Find the probability of extinction if $r = 1/5,\ s = 2/5$ and $t = 2/5$.

[*Standard results on branching processes may be used without proof, provided that they are clearly stated.*]

12F Probability

Planet Zog is a ball with centre O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on its surface. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Spaceships A and B can communicate directly by radio if $\angle AOB < \pi/2$, and similarly for spaceships B and C and spaceships A and C. Given angle $\angle AOB = \gamma < \pi/2$, calculate the probability that C can communicate directly with *either* A or B. Given angle $\angle AOB = \gamma > \pi/2$, calculate the probability that C can communicate directly with *both* A and B. Hence, or otherwise, show that the probability that all three spaceships can keep in in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.