UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 6 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions. If you submit answers to more than four questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, ..., M*** *according to the letter affixed to each question. (For example,* ***23D, 25D*** *should be in one bundle and* ***2J, 6J*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1H Combinatorics

Write an essay on Ramsey theory. You should include the finite and infinite versions of Ramsey's theorem, together with a discussion of upper and lower bounds in the finite case.

[*You may restrict your attention to colourings by just 2 colours.*]

2J Representation Theory

Write an essay on the representation theory of SU_2.

3J Galois Theory

Suppose K, L are fields and $\sigma_1, \ldots, \sigma_m$ are distinct embeddings of K into L. Prove that there do not exist elements $\lambda_1, \ldots, \lambda_m$ of L (not all zero) such that $\lambda_1\sigma_1(x) + \ldots + \lambda_m\sigma_m(x) = 0$ for all $x \in K$. Deduce that if K/k is a finite extension of fields, and $\sigma_1, \ldots, \sigma_m$ are distinct k-automorphisms of K, then $m \leqslant [K : k]$.

Suppose now that K is a Galois extension of k with Galois group cyclic of order n, where n is not divisible by the characteristic. If k contains a primitive nth root of unity, prove that K is a radical extension of k. Explain briefly the relevance of this result to the problem of solubility of cubics by radicals.

4K Differentiable Manifolds

State and prove Stokes' Theorem for compact oriented manifolds-with-boundary.

[*You may assume results relating local forms on the manifold with those on its boundary provided you state them clearly.*]

Deduce that every differentiable map of the unit ball in $\mathbb{R}^n$ to itself has a fixed point.

5J Algebraic Topology

State the Mayer-Vietoris theorem for a finite simplicial complex X which is the union of closed subcomplexes A and B. Define all the maps in the long exact sequence. Prove that the sequence is exact at the term H_iX, for every $i \geqslant 0$.

6J Number Fields

Write an essay on one of the following topics.

(i) Dirichlet's unit theorem and the Pell equation.

(ii) Ideals and the fundamental theorem of arithmetic.

(iii) Dedekind's theorem and the factorisation of primes. (You should treat explicitly either the case of quadratic fields or that of the cyclotomic field.)

7K Hilbert Spaces

Throughout this question, H is an infinite-dimensional, separable Hilbert space. You may use, without proof, any theorems about compact operators that you require.

Define a *Fredholm operator* T, on a Hilbert space H, and define the *index* of T.

(i) Prove that if T is Fredholm then $\operatorname{im} T$ is closed.

(ii) Let $F \in \mathcal{B}(H)$ and let F have finite rank. Prove that F^* also has finite rank.

(iii) Let $T = I + F$, where I is the identity operator on H and F has finite rank; let $E = \operatorname{im} F + \operatorname{im} F^*$. By considering $T|E$ and $T|E^\perp$ (or otherwise) prove that T is Fredholm with $\operatorname{ind} T = 0$.

(iv) Let $T \in \mathcal{B}(H)$ be Fredholm with $\operatorname{ind} T = 0$. Prove that $T = A + F$, where A is invertible and F has finite rank.

[*You may wish to note that T effects an isomorphism from* $(\ker T)^\perp$ *onto* $\operatorname{im} T$*; also* $\ker T$ *and* $(\operatorname{im} T)^\perp$ *have the same finite dimension.*]

(v) Deduce from (iii) and (iv) that $T \in \mathcal{B}(H)$ is Fredholm with $\operatorname{ind} T = 0$ if and only if $T = A + K$ with A invertible and K compact.

(vi) Explain briefly, by considering suitable shift operators on ℓ^2 (i.e. *not* using any theorems about Fredholm operators) that, for each $k \in \mathbb{Z}$, there is a Fredholm operator S on H with $\operatorname{ind} S = k$.

8K Riemann Surfaces

A holomorphic map $p : S \to T$ between Riemann surfaces is called a covering map if every $t \in T$ has a neighbourhood V for which $p^{-1}(V)$ breaks up as a disjoint union of open subsets U_α on which $p : U_\alpha \to V$ is biholomorphic.

(a) Suppose that $f : R \to T$ is any holomorphic map of connected Riemann surfaces, R is simply connected and $p : S \to T$ is a covering map. By considering the lifts of paths from T to S, or otherwise, prove that f lifts to a holomorphic map $\widetilde{f} : R \to S$, i.e. that there exists an $\widetilde{f}$ with $f = p \circ \widetilde{f}$.

(b) Write down a biholomorphic map from the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ onto a half-plane. Show that the unit disk Δ uniformizes the punctured unit disk $\Delta^\times = \Delta - \{0\}$ by constructing an explicit covering map $p : \Delta \to \Delta^\times$.

(c) Using the uniformization theorem, or otherwise, prove that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than one is constant.

9K Algebraic Curves

Write an essay on the Riemann–Roch theorem and some of its applications.

10J Logic, Computation and Set Theory

Explain what is meant by a *well-ordering* of a set.

Without assuming Zorn's Lemma, show that the power-set of any well-ordered set can be given a total (linear) ordering.

By a *selection function* for a set A, we mean a function $f : PA \to PA$ such that $f(B) \subset B$ for all $B \subset A$, $f(B) \neq \emptyset$ for all $B \neq \emptyset$, and $f(B) \neq B$ if B has more than one element. Suppose given a selection function f. Given a mapping $g : \alpha \to [0, 1]$ for some ordinal α, we define a subset $B(f, g) \subset A$ recursively as follows:

$$
\begin{aligned}
&B(f,g) = A \quad \text{if } \alpha = 0, \\
&B(f,g) = f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{ and } g(\beta) = 1, \\
&B(f,g) = B(f,g|_\beta)\backslash f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{and } g(\beta) = 0, \\
&B(f,g) = \bigcap\{B(f,g|_\beta) \mid \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}
\end{aligned}
$$

Show that, for any $x \in A$ and any ordinal α, there exists a function g with domain α such that $x \in B(f, g)$.
[*It may help to observe that g is uniquely determined by x and α, though you need not show this explicitly.*]

Show also that there exists α such that, for every g with domain α, $B(f, g)$ is either empty or a singleton.

Deduce that the assertion 'Every set has a selection function' implies that every set can be totally ordered.
[*Hartogs' Lemma may be assumed, provided you state it precisely.*]

11L Probability and Measure

State Birkhoff's Almost Everywhere Ergodic Theorem for measure-preserving transformations. Define what it means for a sequence of random variables to be *stationary*. Explain *briefly* how the stationarity of a sequence of random variables implies that a particular transformation is measure-preserving.

A bag contains one white ball and one black ball. At each stage of a process one ball is picked from the bag (uniformly at random) and then returned to the bag together with another ball of the same colour. Let X_n be a random variable which takes the value 0 if the nth ball added to the bag is white and 1 if it is black.

(a) Show that the sequence $X_1, X_2, X_3, \ldots$ is stationary and hence that the proportion of black balls in the bag converges almost surely to some random variable R.

(b) Find the distribution of R.

[*The fact that almost-sure convergence implies convergence in distribution may be used without proof.*]

12L Applied Probability

Define a Poisson random measure. State and prove the Product Theorem for the jump times J_n of a Poisson process with constant rate λ and independent random variables Y_n with law μ. Write down the corresponding result for a Poisson process Π in a space $E = \mathbb{R}^d$ with rate $\lambda(x)$ $(x \in E)$ when we associate with each $X \in \Pi$ an independent random variable m_X with density $\rho(X, dm)$.

Prove Campbell's Theorem, i.e. show that if M is a Poisson random measure on the space E with intensity measure ν and $a : E \to \mathbb{R}$ is a bounded measurable function then

$$\mathbf{E}[e^{\theta\Sigma}] = \exp\left(\int_E (e^{\theta a(y)} - 1)\nu(dy)\right),$$

where

$$\Sigma = \int_E a(y)M(dy) = \sum_{X\in\Pi} a(X).$$

Stars are scattered over three-dimensional space $\mathbb{R}^3$ in a Poisson process Π with density $\nu(X)$ $(X \in \mathbb{R}^3)$. Masses of the stars are independent random variables; the mass m_X of a star at X has the density $\rho(X, dm)$. The gravitational potential at the origin is given by

$$F = \sum_{X\in\Pi} \frac{Gm_X}{|X|},$$

where G is a constant. Find the moment generating function $\mathbf{E}[e^{\theta F}]$.

A galaxy occupies a sphere of radius R centred at the origin. The density of stars is $\nu(\mathbf{x}) = 1/|\mathbf{x}|$ for points $\mathbf{x}$ inside the sphere; the mass of each star has the exponential distribution with mean M. Calculate the expected potential due to the galaxy at the origin. Let C be a positive constant. Find the distribution of the distance from the origin to the nearest star whose contribution to the potential F is at least C.

13M Information Theory

Define the Huffman binary encoding procedure and prove its optimality among decipherable codes.

14L Optimization and Control

A discrete-time decision process is defined on a finite set of states I as follows. Upon entry to state i_t at time t the decision-maker observes a variable ξ_t. He then chooses the next state freely within I, at a cost of $c(i_t, \xi_t, i_{t+1})$. Here $\{\xi_0, \xi_1, \ldots\}$ is a sequence of integer-valued, identically distributed random variables. Suppose there exist $\{\phi_i : i \in I\}$ and λ such that for all $i \in I$

$$\phi_i + \lambda = \sum_{k \in \mathbb{Z}} P(\xi_t = k) \min_{i' \in I} \left[c(i, k, i') + \phi_{i'} \right] .$$

Let π denote a policy. Show that

$$\lambda = \inf_{\pi} \limsup_{t \to \infty} E_{\pi} \left[\frac{1}{t} \sum_{s=0}^{t-1} c(i_s, \xi_s, i_{s+1}) \right] .$$

At the start of each month a boat manufacturer receives orders for 1, 2 or 3 boats. These numbers are equally likely and independent from month to month. He can produce j boats in a month at a cost of $6+3j$ units. All orders are filled at the end of the month in which they are ordered. It is possible to make extra boats, ending the month with a stock of i unsold boats, but i cannot be more than 2, and a holding cost of ci is incurred during any month that starts with i unsold boats in stock. Write down an optimality equation that can be used to find the long-run expected average-cost.

Let π be the policy of only ever producing sufficient boats to fill the present month's orders. Show that it is optimal if and only if $c \geqslant 2$.

Suppose $c < 2$. Starting from π, what policy is obtained after applying one step of the policy-improvement algorithm?

15M Principles of Statistics

(a) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from a one-parameter distribution with density function

$$f(x;\theta) = h(x)g(\theta)\exp\{\theta t(x)\}\ ,\ x \in \mathbb{R}.$$

Explain in detail how you would test

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0\ .$$

What is the general form of a conjugate prior density for θ in a Bayesian analysis of this distribution?

(b) Let Y_1, Y_2 be independent Poisson random variables, with means $(1-\psi)\lambda$ and $\psi\lambda$ respectively, with λ *known.*

Explain why the Conditionality Principle leads to inference about ψ being drawn from the conditional distribution of Y_2, given Y_1+Y_2. What is this conditional distribution?

(c) Suppose Y_1, Y_2 have distributions as in (b), but that λ is now *unknown.*

Explain in detail how you would test $H_0 : \psi = \psi_0$ against $H_1 : \psi \neq \psi_0$, and describe the optimality properties of your test.

[*Any general results you use should be stated clearly, but need not be proved.*]

16L Stochastic Financial Models

Write an essay on the Black–Scholes formula for the price of a European call option on a stock. Your account should include a derivation of the formula and a careful analysis of its dependence on the parameters of the model.

17F Dynamical Systems

Let $\mathcal{S}$ be a metric space, F a map of $\mathcal{S}$ to itself and P a point of $\mathcal{S}$. Define an *attractor* for F and an *omega point* of the orbit of P under F.

Let f be the map of $\mathbb{R}$ to itself given by

$$f(x) = x + \frac{1}{2} + c\sin^2 2\pi x,$$

where $c > 0$ is so small that $f'(x) > 0$ for all x, and let F be the map of $\mathbb{R}/\mathbb{Z}$ to itself induced by f. What points if any are
(a) attractors for F^2,
(b) omega points of the orbit of some point P under F ?
Is the cycle $\{0,\ \frac{1}{2}\}$ an attractor?

In the notation of the first two sentences, let $\mathcal{C}$ be a cycle of order M and assume that F is continuous. Prove that $\mathcal{C}$ is an attractor for F if and only if each point of $\mathcal{C}$ is an attractor for F^M.

18G Partial Differential Equations

Discuss the notion of *fundamental solution* for a linear partial differential equation with constant coefficients.

19G Methods of Mathematical Physics

Let

$$I(\lambda, a) = \int_{-i\infty}^{i\infty} \frac{e^{\lambda(t^3-3t)}}{t^2 - a^2}\, dt\ ,$$

where λ is real, a is real and non-zero, and the path of integration runs up the imaginary axis. Show that, if $a^2 > 1$,

$$I(\lambda, a) \sim \frac{ie^{-2\lambda}}{1-a^2}\sqrt{\frac{\pi}{3\lambda}}$$

as $\lambda \to +\infty$ and sketch the relevant steepest descent path.

What is the corresponding result if $a^2 < 1$?

20F Numerical Analysis

Write an essay on the method of conjugate gradients. You should describe the algorithm, present an analysis of its properties and discuss its advantages.

[*Any theorems quoted should be stated precisely but need not be proved.*]

21E Electrodynamics

Derive Larmor's formula for the rate at which radiation is produced by a particle of charge q moving along a trajectory $\mathbf{x}(t)$.

A non-relativistic particle of mass m, charge q and energy E is incident along a radial line in a central potential $V(r)$. The potential is vanishingly small for r very large, but increases without bound as $r \to 0$. Show that the total amount of energy $\mathcal{E}$ radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}}\left(\frac{dV}{dr}\right)^2 dr\,,$$

where $V(r_0) = E$.

Suppose that V is the Coulomb potential $V(r) = A/r$. Evaluate $\mathcal{E}$.

22E Foundations of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$-\frac{2e^2}{4\pi\epsilon_0 r}$$

has normalized, spherically symmetric, real wave functions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the ground state of the helium atom? Assuming that wavefunctions which are not spherically symmetric can be ignored, what are the states of the first excited energy level of the helium atom?
[*You may assume here that the electrons are non-interacting.*]

Show that, taking into account the interaction between the two electrons, the estimate for the energy of the ground state of the helium atom is

$$2E_0 + \frac{e^2}{4\pi\epsilon_0}\int \frac{d^3\mathbf{r}_1\, d^3\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\psi_0^2(\mathbf{r}_1)\psi_0^2(\mathbf{r}_2).$$

23D Statistical Physics

A perfect gas in equilibrium in a volume V has quantum stationary states $|i\rangle$ with energies E_i. In a Boltzmann distribution, the probability that the system is in state $|i\rangle$ is $\rho_i = Z^{-1} e^{-E_i/kT}$. The entropy is defined to be $S = -k \sum_i \rho_i \log \rho_i$.

For two nearby states establish the equation

$$dE \;=\; TdS \;-\; PdV \;,$$

where E and P should be defined.

For reversible changes show that

$$dS \;=\; \frac{\delta Q}{T} \;,$$

where δQ is the amount of heat transferred in the exchange.

Define C_V, the heat capacity at constant volume.

A system with constant heat capacity C_V initially at temperature T is heated at constant volume to a temperature Θ. Show that the change in entropy is $\Delta S = C_V \log(\Theta/T)$.

Explain what is meant by isothermal and adiabatic transitions.

Briefly, describe the Carnot cycle and define its efficiency. Explain briefly why no heat engine can be more efficient than one whose operation is based on a Carnot cycle.

Three identical bodies with constant heat capacity at fixed volume C_V, are initially at temperatures T_1, T_2, T_3, respectively. Heat engines operate between the bodies with no input of work or heat from the outside and the respective temperatures are changed to $\Theta_1, \Theta_2, \Theta_3$, the volume of the bodies remaining constant. Show that, if the heat engines operate on a Carnot cycle, then

$$\Theta_1 \Theta_2 \Theta_3 \;=\; A \;, \qquad \Theta_1 + \Theta_2 + \Theta_3 \;=\; B \;,$$

where $A = T_1 T_2 T_3$ and $B = T_1 + T_2 + T_3$.

Hence show that the maximum temperature to which any one of the bodies can be raised is Θ where

$$\Theta \;+\; 2\left(\frac{A}{\Theta}\right)^{1/2} \;=\; B \;.$$

Show that a solution is $\Theta = T$ if initially $T_1 = T_2 = T_3 = T$. Do you expect there to be any other solutions?

Find Θ if initially $T_1 = 300\,\mathrm{K},\; T_2 = 300\,\mathrm{K},\; T_3 = 100\,\mathrm{K}$.

[*Hint: Choose to maximize one temperature and impose the constraints above using Lagrange multipliers.*]

24D Applications of Quantum Mechanics

Explain the variational method for computing the ground state energy for a quantum Hamiltonian.

For the one-dimensional Hamiltonian

$$H = \tfrac{1}{2}p^2 + \lambda x^4 \,,$$

obtain an approximate form for the ground state energy by considering as a trial state the state $|w\rangle$ defined by $a|w\rangle = 0$, where $\langle w|w\rangle = 1$ and $a = (w/2\hbar)^{\frac{1}{2}}(x + ip/w)$.

[*It is useful to note that* $\langle w|(a + a^\dagger)^4|w\rangle = \langle w|(a^2 a^{\dagger 2} + aa^\dagger aa^\dagger)|w\rangle$.]

Explain why the states $a^\dagger|w\rangle$ may be used as trial states for calculating the first excited energy level.

25D General Relativity

With respect to the Schwarzschild coordinates (r, θ, ϕ, t), the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(1 - \frac{r_s}{r}\right) dt^2,$$

where $r_s = 2M$ is the Schwarzschild radius and M is the Schwarzschild mass. Show that, by a suitable choice of (θ, ϕ), the general geodesic can regarded as moving in the equatorial plane $\theta = \pi/2$. Obtain the equations governing timelike and null geodesics in terms of $u(\phi)$, where $u = 1/r$.

Discuss light bending and perihelion precession in the solar system.

26C Fluid Dynamics II

Write an essay on boundary-layer theory and its application to the generation of lift in aerodynamics.

You should include discussion of the derivation of the boundary-layer equation, the similarity transformation leading to the Falkner–Skan equation, the influence of an adverse pressure gradient, and the mechanism(s) by which circulation is generated in flow past bodies with a sharp trailing edge.

27C Waves in Fluid and Solid Media

Write down the equation governing linearized displacements $\mathbf{u}(\mathbf{x}, t)$ in a uniform elastic medium of density ρ and Lamé constants λ and μ. Derive solutions for monochromatic plane P and S waves, and find the corresponding wave speeds c_P and c_S.

Such an elastic solid occupies the half-space $z > 0$, and the boundary $z = 0$ is clamped rigidly so that $\mathbf{u}(x, y, 0, t) = \mathbf{0}$. A plane SV-wave with frequency ω and wavenumber $(k, 0, -m)$ is incident on the boundary. At some angles of incidence, there results both a reflected SV-wave with frequency ω' and wavenumber $(k', 0, m')$ and a reflected P-wave with frequency ω'' and wavenumber $(k'', 0, m'')$. Relate the frequencies and wavenumbers of the reflected waves to those of the incident wave. At what angles of incidence will there be a reflected P-wave?

Find the amplitudes of the reflected waves as multiples of the amplitude of the incident wave. Confirm that these amplitudes give the sum of the time-averaged vertical fluxes of energy of the reflected waves equal to the time-averaged vertical flux of energy of the incident wave.

[*Results concerning the energy flux, energy density and kinetic energy density in a plane elastic wave may be quoted without proof.*]