UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 5 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, ..., H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

Suppose that for each $n = 1, 2, \ldots$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) If $f_n \to f$ pointwise on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

(b) If $f_n \to f$ uniformly on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

In each case, give a proof or a counter-example (with justification).

2A Methods

Find the Fourier sine series for $f(x) = x$, on $0 \leqslant x < L$. To which value does the series converge at $x = \frac{3}{2}L$?

Now consider the corresponding cosine series for $f(x) = x$, on $0 \leqslant x < L$. Sketch the cosine series between $x = -2L$ and $x = 2L$. To which value does the series converge at $x = \frac{3}{2}L$? [*You do not need to determine the cosine series explicitly.*]

3H Statistics

State the factorization criterion for sufficient statistics and give its proof in the discrete case.

Let $X_1, \ldots, X_n$ form a random sample from a Poisson distribution for which the value of the mean θ is unknown. Find a one-dimensional sufficient statistic for θ.

4E Geometry

Show that any finite group of orientation-preserving isometries of the Euclidean plane is cyclic.

Show that any finite group of orientation-preserving isometries of the hyperbolic plane is cyclic.

[*You may assume that given any non-empty finite set E in the hyperbolic plane, or the Euclidean plane, there is a unique smallest closed disc that contains E. You may also use any general fact about the hyperbolic plane without proof providing that it is stated carefully.*]

5G Linear Mathematics

Define $f : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(a, b, c) = (a + 3b - c, 2b + c, -4b - c).$$

Find the characteristic polynomial and the minimal polynomial of f. Is f diagonalisable? Are f and f^2 linearly independent endomorphisms of $\mathbb{C}^3$? Justify your answers.

6C Fluid Dynamics

A fluid flow has velocity given in Cartesian co-ordinates as $\mathbf{u} = (kty, 0, 0)$ where k is a constant and t is time. Show that the flow is incompressible. Find a stream function and determine an equation for the streamlines at time t.

At $t = 0$ the points along the straight line segment $x = 0,\ 0 \leqslant y \leqslant a,\ z = 0$ are marked with dye. Show that at any later time the marked points continue to form a segment of a straight line. Determine the length of this line segment at time t and the angle that it makes with the x-axis.

7B Complex Methods

Using contour integration around a rectangle with vertices

$$-x,\ x,\ x + iy,\ -x + iy,$$

prove that, for all real y,

$$\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx\,.$$

Hence derive that the function $f(x) = e^{-x^2/2}$ is an eigenfunction of the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx,$$

i.e. $\widehat{f}$ is a constant multiple of f.

8F Quadratic Mathematics

Define the *rank* and *signature* of a symmetric bilinear form ϕ on a finite-dimensional real vector space. (If your definitions involve a matrix representation of ϕ, you should explain why they are independent of the choice of representing matrix.)

Let V be the space of all $n \times n$ real matrices (where $n \geqslant 2$), and let ϕ be the bilinear form on V defined by

$$\phi(A, B) = \operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B \ .$$

Find the rank and signature of ϕ.

[*Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.*]

9D Quantum Mechanics

Consider a quantum mechanical particle of mass m moving in one dimension, in a potential well

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 < x < a, \\ V_0, & x > a. \end{cases}$$

Sketch the ground state energy eigenfunction $\chi(x)$ and show that its energy is $E = \frac{\hbar^2 k^2}{2m}$, where k satisfies

$$\tan ka = -\frac{k}{\sqrt{\frac{2mV_0}{\hbar^2} - k^2}}.$$

[*Hint: You may assume that* $\chi(0) = 0$.]

SECTION II

10E Analysis II

Suppose that (X, d) is a metric space that has the Bolzano-Weierstrass property (that is, any sequence has a convergent subsequence). Let (Y, d') be any metric space, and suppose that f is a continuous map of X onto Y. Show that (Y, d') also has the Bolzano-Weierstrass property.

Show also that if f is a bijection of X onto Y, then $f^{-1} : Y \to X$ is continuous.

By considering the map $x \mapsto e^{ix}$ defined on the real interval $[-\pi/2, \pi/2]$, or otherwise, show that there exists a continuous choice of $\arg z$ for the complex number z lying in the right half-plane $\{x + iy : x > 0\}$.

11A Methods

The potential $\Phi(r, \vartheta)$, satisfies Laplace's equation everywhere except on a sphere of unit radius and $\Phi \to 0$ as $r \to \infty$. The potential is continuous at $r = 1$, but the derivative of the potential satisfies

$$\lim_{r \to 1^+} \frac{\partial \Phi}{\partial r} - \lim_{r \to 1^-} \frac{\partial \Phi}{\partial r} = V \cos^2 \vartheta,$$

where V is a constant. Use the method of separation of variables to find Φ for both $r > 1$ and $r < 1$.

[*The Laplacian in spherical polar coordinates for axisymmetric systems is*

$$\nabla^2 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\left(\frac{\partial}{\partial \vartheta} \sin\vartheta \frac{\partial}{\partial \vartheta}\right).$$

You may assume that the equation

$$\big((1 - x^2) y'\big)' + \lambda y = 0$$

has polynomial solutions of degree n, which are regular at $x = \pm 1$, if and only if $\lambda = n(n+1)$.]

12H Statistics

Suppose we ask 50 men and 150 women whether they are early risers, late risers, or risers with no preference. The data are given in the following table.

	Early risers	*Late risers*	*No preference*	*Totals*
Men	17	22	11	50
Women	43	78	29	150
Totals	60	100	40	200

Derive carefully a (generalized) likelihood ratio test of independence of classification. What is the result of applying this test at the 0.01 level?

[*Distribution* χ_1^2 χ_2^2 χ_3^2 χ_5^2 χ_6^2
99%*percentile* *6.63* *9.21* *11.34* *15.09* *16.81*]

13E Geometry

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, and let $\mathbb{H}$ have the hyperbolic metric ρ derived from the line element $|dz|/y$. Let Γ be the group of Möbius maps of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c and d are real and $ad - bc = 1$. Show that every g in Γ is an isometry of the metric space $(\mathbb{H}, \rho)$. For z and w in $\mathbb{H}$, let

$$h(z,w) = \frac{|z-w|^2}{\operatorname{Im}(z)\operatorname{Im}(w)}.$$

Show that for every g in Γ, $h\big(g(z), g(w)\big) = h(z,w)$. By considering $z = iy$, where $y > 1$, and $w = i$, or otherwise, show that for all z and w in $\mathbb{H}$,

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\operatorname{Im}(z)\operatorname{Im}(w)}.$$

By considering points i, iy, where $y > 1$ and $s + it$, where $s^2 + t^2 = 1$, or otherwise, derive Pythagoras' Theorem for hyperbolic geometry in the form $\cosh a \cosh b = \cosh c$, where a, b and c are the lengths of sides of a right-angled triangle whose hypotenuse has length c.

14G Linear Mathematics

Let α be an endomorphism of a vector space V of finite dimension n .

(a) What is the dimension of the vector space of linear endomorphisms of V? Show that there exists a non-trivial polynomial $p(X)$ such that $p(\alpha) = 0$. Define what is meant by the minimal polynomial m_α of α.

(b) Show that the eigenvalues of α are precisely the roots of the minimal polynomial of α.

(c) Let W be a subspace of V such that $\alpha(W) \subseteq W$ and let β be the restriction of α to W. Show that m_β divides m_α.

(d) Give an example of an endomorphism α and a subspace W as in (c) not equal to V for which $m_\alpha = m_\beta$, and $\deg(m_\alpha) > 1$.

15C Fluid Dynamics

State the unsteady form of Bernoulli's theorem.

A spherical bubble having radius R_0 at time $t = 0$ is located with its centre at the origin in unbounded fluid. The fluid is inviscid, has constant density ρ and is everywhere at rest at $t = 0$. The pressure at large distances from the bubble has the constant value p_∞, and the pressure inside the bubble has the constant value $p_\infty - \triangle p$. In consequence the bubble starts to collapse so that its radius at time t is $R(t)$. Find the velocity everywhere in the fluid in terms of $R(t)$ at time t and, assuming that surface tension is negligible, show that R satisfies the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{\triangle p}{\rho} \ .$$

Find the total kinetic energy of the fluid in terms of $R(t)$ at time t. Hence or otherwise obtain a first integral of the above equation.

16B Complex Methods

(a) Show that if f is an analytic function at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0, i.e. it preserves angles between paths passing through z_0.

(b) Let D be the disc given by $|z + i| < \sqrt{2}$, and let H be the half-plane given by $y > 0$, where $z = x + iy$. Construct a map of the domain $D \cap H$ onto H, and hence find a conformal mapping of $D \cap H$ onto the disc $\{z : |z| < 1\}$. [*Hint: You may find it helpful to consider a mapping of the form* $(az + b)/(cz + d)$, *where* $ad - bc \neq 0$.]

17F Quadratic Mathematics

Let A and B be $n \times n$ real symmetric matrices, such that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite. Show that it is possible to find an invertible matrix P such that $P^T AP = I$ and $P^T BP$ is diagonal. Show also that the diagonal entries of the matrix $P^T BP$ may be calculated directly from A and B, without finding the matrix P. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

find the diagonal entries of $P^T BP$.

18D Quantum Mechanics

A quantum mechanical particle of mass M moves in one dimension in the presence of a negative delta function potential

$$V = -\frac{\hbar^2}{2M\Delta}\delta(x),$$

where Δ is a parameter with dimensions of length.

(a) Write down the time-independent Schrödinger equation for energy eigenstates $\chi(x)$, with energy E. By integrating this equation across $x = 0$, show that the gradient of the wavefunction jumps across $x = 0$ according to

$$\lim_{\epsilon \to 0} \left(\frac{d\chi}{dx}(\epsilon) - \frac{d\chi}{dx}(-\epsilon) \right) \;=\; -\frac{1}{\Delta}\chi(0).$$

[*You may assume that χ is continuous across $x = 0$.*]

(b) Show that there exists a negative energy solution and calculate its energy.

(c) Consider a double delta function potential

$$V(x) = -\frac{\hbar^2}{2M\Delta}[\delta(x + a) + \delta(x - a)].$$

For sufficiently small Δ, this potential yields a negative energy solution of odd parity, i.e. $\chi(-x) = -\chi(x)$. Show that its energy is given by

$$E = -\frac{\hbar^2}{2M}\lambda^2, \quad \text{where} \quad \tanh \lambda a = \frac{\lambda\Delta}{1 - \lambda\Delta}.$$

[*You may again assume χ is continuous across $x = \pm a$.*]

END OF PAPER