UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 4 June 2002 1.30 - 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in three separate bundles, marked ***A***, ***B*** *and* ***E*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet <u>*must*</u> ***bear your examination number and desk number.***

SECTION I

1A Algebra and Geometry

Given two real non-zero 2×2 matrices A and B, with $AB = 0$, show that A maps $\mathbb{R}^2$ onto a line. Is it always true that $BA = 0$? Show that there is always a non-zero matrix C with $CA = 0 = AC$. Justify your answers.

2B Algebra and Geometry

(a) What does it mean for a group to be cyclic? Give an example of a finite abelian group that is not cyclic, and justify your assertion.

(b) Suppose that G is a finite group of rotations of $\mathbb{R}^2$ about the origin. Is G necessarily cyclic? Justify your answer.

3A Vector Calculus

Determine whether each of the following is the exact differential of a function, and if so, find such a function:

(a) $(\cosh\theta + \sinh\theta\cos\phi)d\theta + (\cosh\theta\sin\phi + \cos\phi)d\phi$,

(b) $3x^2(y^2+1)dx + 2(yx^3 - z^2)dy - 4yzdz$.

4A Vector Calculus

State the divergence theorem.

Consider the integral

$$I = \int_S r^n \mathbf{r} \cdot d\mathbf{S} \ ,$$

where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

SECTION II

5E Algebra and Geometry

Prove, using the standard formula connecting δ_{ij} and ϵ_{ijk}, that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \,.$$

Define, in terms of the dot and cross product, the triple scalar product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ and show that it is invariant under cyclic permutation of the vectors.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a *not necessarily orthonormal* basis for $\mathbb{R}^3$, and define

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,.$$

By calculating $[\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]$, show that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is also a basis for $\mathbb{R}^3$.

The vectors $\hat{\hat{\mathbf{e}}}_1, \hat{\hat{\mathbf{e}}}_2, \hat{\hat{\mathbf{e}}}_3$ are constructed from $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the same way that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are constructed from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Show that

$$\hat{\hat{\mathbf{e}}}_1 = \mathbf{e}_1, \;\; \hat{\hat{\mathbf{e}}}_2 = \mathbf{e}_2, \;\; \hat{\hat{\mathbf{e}}}_3 = \mathbf{e}_3,$$

Show that a vector $\mathbf{V}$ has components $\mathbf{V} \cdot \hat{\mathbf{e}}_1, \; \mathbf{V} \cdot \hat{\mathbf{e}}_2, \; \mathbf{V} \cdot \hat{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. What are the components of the vector $\mathbf{V}$ with respect to the basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$?

6E Algebra and Geometry

(a) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a} \,, \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}) \,.$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be specified.

(b) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

7A Algebra and Geometry

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is $0, 1$ or infinite, where A is a real 3×3 matrix and $\mathbf{b} \in \mathbb{R}^3$. Let α be the mapping which A represents. State necessary and sufficient conditions on $\mathbf{b}$ and α for each of these possibilities to hold.

Let A and B be 3×3 matrices representing linear mappings α and β. Give necessary and sufficient conditions on α and β for the existence of a 3×3 matrix X with $AX = B$. When is X unique?

Find X when

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

8B Algebra and Geometry

Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of a regular tetrahedron T in $\mathbb{R}^3$ and that $\mathbf{a} = (1,1,1)$, $\mathbf{b} = (-1,-1,1)$, $\mathbf{c} = (-1,1,-1)$, $\mathbf{d} = (1,x,y)$.

(a) Find x and y.

(b) Find a matrix M that is a rotation leaving T invariant such that $M\mathbf{a} = \mathbf{b}$ and $M\mathbf{b} = \mathbf{a}$.

9A Vector Calculus

Two independent variables x_1 and x_2 are related to a third variable t by

$$x_1 = a + \alpha t\ , \quad x_2 = b + \beta t\ ,$$

where a, b, α and β are constants. Let f be a smooth function of x_1 and x_2, and let $F(t) = f(x_1, x_2)$. Show, by using the Taylor series for $F(t)$ about $t = 0$, that

$$f(x_1, x_2) = f(a, b) + (x_1 - a)\frac{\partial f}{\partial x_1} + (x_2 - b)\frac{\partial f}{\partial x_2}$$
$$+\frac{1}{2}\left((x_1 - a)^2\frac{\partial^2 f}{\partial x_1^2} + 2(x_1 - a)(x_2 - b)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2 - b)^2\frac{\partial^2 f}{\partial x_2^2}\right) + \ldots,$$

where all derivatives are evaluated at $x_1 = a,\ x_2 = b$.

Hence show that a stationary point (a, b) of $f(x_1, x_2)$ is a local minimum if

$$H_{11} > 0, \quad \det H_{ij} > 0\ ,$$

where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ is the Hessian matrix evaluated at (a, b).

Find two local minima of

$$f(x_1, x_2) = x_1^4 - x_1^2 + 2x_1x_2 + x_2^2\ .$$

10A Vector Calculus

The domain S in the (x, y) plane is bounded by $y = x$, $y = ax\,(0 \leqslant a \leqslant 1)$ and $xy^2 = 1\,(x, y \geqslant 0)$. Find a transformation

$$u = f(x, y), \quad v = g(x, y)\ ,$$

such that S is transformed into a rectangle in the (u, v) plane.

Evaluate

$$\int_D \frac{y^2 z^2}{x}\, dx\, dy\, dz\ ,$$

where D is the region bounded by

$$y = x\ , \qquad y = zx, \qquad xy^2 = 1 \qquad (x, y \geqslant 0)$$

and the planes

$$z = 0\,, \qquad z = 1\,.$$

11A Vector Calculus

Prove that

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}.$$

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n} \cdot \mathbf{v} = 0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S \left(\delta_{ij} - n_i n_j\right) \frac{\partial v_i}{\partial x_j}\, dS = \oint_C \mathbf{u} \cdot \mathbf{v}\, ds\ ,$$

where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}ds = d\mathbf{s} \times \mathbf{n}$. Verify this result by taking $\mathbf{v} = \mathbf{r}$, and S to be the disc $|\mathbf{r}| \leqslant R$ in the $z = 0$ plane.

12A Vector Calculus

(a) Show, using Cartesian coordinates, that $\psi = 1/r$ satisfies Laplace's equation, $\nabla^2\psi = 0$, on $\mathbb{R}^3 \backslash \{0\}$.

(b) ϕ and ψ are smooth functions defined in a 3-dimensional domain V bounded by a smooth surface S. Show that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{S}.$$

(c) Let $\psi = 1/|\mathbf{r} - \mathbf{r}_0|$, and let V_ε be a domain bounded by a smooth outer surface S and an inner surface S_ε, where S_ε is a sphere of radius ε, centre $\mathbf{r}_0$. The function ϕ satisfies

$$\nabla^2\phi = -\rho(\mathbf{r}).$$

Use parts (a) and (b) to show, taking the limit $\varepsilon \to 0$, that ϕ at $\mathbf{r}_0$ is given by

$$4\pi\phi(\mathbf{r}_0) = \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} dV + \int_S \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \frac{\partial\phi}{\partial n} - \phi(\mathbf{r}) \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) dS,$$

where V is the domain bounded by S.

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