UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

List of Courses

1/I/5C Linear Mathematics

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix} x & 1 & 1 \\ 1-x & 0 & -1 \\ 2 & 2x & 1 \end{pmatrix}$$

is invertible. Determine the rank of M as a function of x. Find the adjugate and hence the inverse of M for general x.

1/II/14C Linear Mathematics

(a) Find a matrix M over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x-2)^3(x+1)^2$. Furthermore find two matrices M_1 and M_2 over $\mathbb{C}$ which have the same characteristic polynomial, $(x-3)^5(x-1)^2$, and the same minimal polynomial, $(x-3)^2(x-1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, M_3, neither conjugate to M_1 nor to M_2? Justify your answer.

(b) Suppose A is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x-\lambda_1)(x-\lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which A defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1 \iota) \oplus \ker(\alpha - \lambda_2 \iota)$, where ι is the identity.

[*Hint: Express* $v \in V$ *in terms of* $(\alpha - \lambda_1\iota)(v)$ *and* $(\alpha - \lambda_2\iota)(v)$.]

Now suppose that A has minimal polynomial $(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_m)$ for distinct $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \ldots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

2/I/6C Linear Mathematics

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$. Find the matrix representing ρ_A with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2\times 2}(\mathbb{C})$. Prove that the characteristic polynomial of ρ_A is equal to the square of the characteristic polynomial of A, and that A and ρ_A have the same minimal polynomial.

2/II/15C Linear Mathematics

Define the dual V^* of a vector space V. Given a basis $\{v_1, \ldots, v_n\}$ of V define its dual and show it is a basis of V^*. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of V and W relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of V^* and W^*.

Prove that α and α^* have the same rank.

Suppose that α is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

3/I/7C Linear Mathematics

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix M which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is W and which contains $(1,1,1,1,1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel, and image contained in W?

3/II/17C Linear Mathematics

Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of V, i.e. $\alpha^m = 0$ for some positive integer m. Prove that α can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). [*You may wish to consider the subspaces* $\ker(\alpha^j)$ *for* $j = 1, \ldots, m$.]

Show that if α is nilpotent, then $\alpha^n = 0$ where n is the dimension of V. Give an example of a 4×4 matrix M such that $M^4 = 0$ but $M^3 \neq 0$.

Let A be a nilpotent matrix and I the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I+A)(I+B)$ if A and B are nilpotent? Justify your answer.

4/I/6C **Linear Mathematics**

Find the Jordan normal form J of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} ,$$

and determine both the characteristic and the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J (the Jordan normal form of M) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

4/II/15C **Linear Mathematics**

Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that AB and BA have the same characteristic polynomial. [*Hint: Look at* $\det(CBC - xC)$ *for* $C = A + yI$*, where* x *and* y *are scalar variables.*]

Show by example that AB and BA need not have the same minimal polynomial.

Suppose that AB is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of BA must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.

1/I/4B Geometry

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. What are the geodesics passing through the origin? Show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$.

Write down an isometry between the upper half-plane model H of the hyperbolic plane and the disc model Δ, under which $i \in H$ corresponds to $0 \in \Delta$. Show that the hyperbolic circle of radius ρ centred on i in H is a Euclidean circle with centre $i \cosh \rho$ and of radius $\sinh \rho$.

1/II/13B Geometry

Describe geometrically the stereographic projection map ϕ from the unit sphere S^2 to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and find a formula for ϕ. Show that any rotation of S^2 about the z-axis corresponds to a Möbius transformation of $\mathbb{C}_\infty$. You are given that the rotation of S^2 defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

corresponds under ϕ to a Möbius transformation of $\mathbb{C}_\infty$; deduce that any rotation of S^2 about the x-axis also corresponds to a Möbius transformation.

Suppose now that $u, v \in \mathbb{C}$ correspond under ϕ to distinct points $P, Q \in S^2$, and let d denote the angular distance from P to Q on S^2. Show that $-\tan^2(d/2)$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in some order (which you should specify). [*You may assume that the cross-ratio is invariant under Möbius transformations.*]

3/I/4B Geometry

State and prove the Gauss–Bonnet theorem for the area of a spherical triangle.

Suppose $\mathbf{D}$ is a regular dodecahedron, with centre the origin. Explain how each face of $\mathbf{D}$ gives rise to a spherical pentagon on the 2-sphere S^2. For each such spherical pentagon, calculate its angles and area.

3/II/14B **Geometry**

Describe the hyperbolic lines in the upper half-plane model H of the hyperbolic plane. The group $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts on H via Möbius transformations, which you may assume are isometries of H. Show that G acts transitively on the hyperbolic lines. Find explicit formulae for the reflection in the hyperbolic line L in the cases (i) L is a vertical line $x = a$, and (ii) L is the unit semi-circle with centre the origin. Verify that the composite T of a reflection of type (ii) followed afterwards by one of type (i) is given by $T(z) = -z^{-1} + 2a$.

Suppose now that L_1 and L_2 are distinct hyperbolic lines in the hyperbolic plane, with R_1, R_2 denoting the corresponding reflections. By considering different models of the hyperbolic plane, or otherwise, show that

(a) R_1R_2 has infinite order if L_1 and L_2 are parallel or ultraparallel, and

(b) R_1R_2 has finite order if and only if L_1 and L_2 meet at an angle which is a rational multiple of π.

1/I/1A **Analysis II**

Define uniform continuity for functions defined on a (bounded or unbounded) interval in $\mathbb{R}$.

Is it true that a real function defined and uniformly continuous on $[0,1]$ is necessarily bounded?

Is it true that a real function defined and with a bounded derivative on $[1,\infty)$ is necessarily uniformly continuous there?

Which of the following functions are uniformly continuous on $[1,\infty)$:

(i) x^2;

(ii) $\sin(x^2)$;

(iii) $\frac{\sin x}{x}$?

Justify your answers.

1/II/10A **Analysis II**

Show that each of the functions below is a metric on the set of functions $x(t) \in C[a,b]$:

$$d_1(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

$$d_2(x,y) = \left\{ \int_a^b |x(t) - y(t)|^2 dt \right\}^{1/2}.$$

Is the space complete in the d_1 metric? Justify your answer.

Show that the set of functions

$$x_n(t) = \begin{cases} 0, & -1 \leqslant t < 0 \\ nt, & 0 \leqslant t < 1/n \\ 1, & 1/n \leqslant t \leqslant 1 \end{cases}$$

is a Cauchy sequence with respect to the d_2 metric on $C[-1,1]$, yet does not tend to a limit in the d_2 metric in this space. Hence, deduce that this space is not complete in the d_2 metric.

2/I/1A Analysis II

State and prove the contraction mapping theorem.

Let $A = \{x, y, z\}$, let d be the discrete metric on A, and let d' be the metric given by: d' is symmetric and

$$d'(x, y) = 2,\ d'(x, z) = 2,\ d'(y, z) = 1,$$

$$d'(x, x) = d'(y, y) = d'(z, z) = 0.$$

Verify that d' is a metric, and that it is Lipschitz equivalent to d.

Define an appropriate function $f : A \to A$ such that f is a contraction in the d' metric, but not in the d metric.

2/II/10A Analysis II

Define total boundedness for metric spaces.

Prove that a metric space has the Bolzano–Weierstrass property if and only if it is complete and totally bounded.

3/I/1A Analysis II

Define what is meant by a norm on a real vector space.

(a) Prove that two norms on a vector space (not necessarily finite-dimensional) give rise to equivalent metrics if and only if they are Lipschitz equivalent.

(b) Prove that if the vector space V has an inner product, then for all $x, y \in V$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

in the induced norm.

Hence show that the norm on $\mathbb{R}^2$ defined by $\|x\| = \max(|x_1|, |x_2|)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, cannot be induced by an inner product.

3/II/11A **Analysis II**

Prove that if all the partial derivatives of $f : \mathbb{R}^p \to \mathbb{R}$ (with $p \geqslant 2$) exist in an open set containing $(0, 0, \ldots, 0)$ and are continuous at this point, then f is differentiable at $(0, 0, \ldots, 0)$.

Let

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$f(x, y) = g(x) + g(y).$$

At which points of the plane is the partial derivative f_x continuous?

At which points is the function $f(x, y)$ differentiable? Justify your answers.

4/I/1A **Analysis II**

Let f be a mapping of a metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ for all distinct x, y in X.

Show that $f(x)$ and $d(x, f(x))$ are continuous functions of x.

Now suppose that (X, d) is compact and let

$$h = \inf_{x \in X} d(x, f(x)).$$

Show that we cannot have $h > 0$.

[*You may assume that a continuous function on a compact metric space is bounded and attains its bounds.*]

Deduce that f possesses a fixed point, and that it is unique.

4/II/10A **Analysis II**

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions on a closed interval $[a, b]$. Prove that, if for every $x \in [a, b]$, the sequence $\{f_n(x)\}$ is monotonic in n, and if all the functions f_n, $n = 1, 2, \ldots$, and $f = \lim f_n$ are continuous, then $f_n \to f$ uniformly on $[a, b]$.

By considering a suitable sequence of functions $\{f_n\}$ on $[0, 1)$, show that if the interval is not closed but all other conditions hold, the conclusion of the theorem may fail.

1/I/7E Complex Methods

State the Cauchy integral formula.

Assuming that the function $f(z)$ is analytic in the disc $|z| < 1$, prove that, for every $0 < r < 1$, it is true that

$$\frac{d^n f(0)}{dz^n} = \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, \qquad n = 0, 1, \ldots .$$

[*Taylor's theorem may be used if clearly stated.*]

1/II/16E Complex Methods

Let the function F be integrable for all real arguments x, such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \ ,$$

and assume that the series

$$f(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly for all $0 \leqslant \tau \leqslant 2\pi$.

Prove the Poisson summation formula

$$f(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{in\tau},$$

where $\hat{F}$ is the Fourier transform of F. [*Hint: You may show that*

$$\frac{1}{2\pi} \int_{0}^{2\pi} e^{-imx} f(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imx} F(x) dx$$

or, alternatively, prove that f is periodic and express its Fourier expansion coefficients explicitly in terms of $\hat{F}$.]

Letting $F(x) = e^{-|x|}$, use the Poisson summation formula to evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \ .$$

2/I/7E Complex Methods

A complex function is defined for every $z \in V$, where V is a non-empty open subset of $\mathbb{C}$, and it possesses a derivative at every $z \in V$. Commencing from a formal definition of derivative, deduce the Cauchy–Riemann equations.

2/II/16E Complex Methods

Let R be a rational function such that $\lim_{z\to\infty}\{zR(z)\} = 0$. Assuming that R has no real poles, use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} R(x)dx.$$

Given that $n \geqslant 1$ is an integer, evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}.$$

4/I/8F Complex Methods

Consider a conformal mapping of the form

$$f(z) = \frac{a+bz}{c+dz}, \quad z \in \mathbb{C} \ ,$$

where $a, b, c, d \in \mathbb{C}$, and $ad \neq bc$. You may assume $b \neq 0$. Show that any such $f(z)$ which maps the unit circle onto itself is necessarily of the form

$$f(z) = e^{i\psi}\,\frac{a+z}{1+\bar{a}z}.$$

[*Hint: Show that it is always possible to choose* $b = 1$.]

4/II/17F Complex Methods

State Jordan's Lemma.

Consider the integral

$$I = \oint_C dz \frac{z\ \sin(xz)}{(a^2+z^2)\sin \pi z},$$

for real x and a. The rectangular contour C runs from $+\infty + i\epsilon$ to $-\infty + i\epsilon$, to $-\infty - i\epsilon$, to $+\infty - i\epsilon$ and back to $+\infty + i\epsilon$, where ϵ is infinitesimal and positive. Perform the integral in two ways to show that

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin nx}{a^2+n^2} = -\pi \frac{\sinh ax}{\sinh a\pi},$$

for $|x| < \pi$.

1/I/2H **Methods**

The even function $f(x)$ has the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $-\pi \leqslant x \leqslant \pi$. Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2.$$

Find the Fourier cosine series of x^2 in the same interval, and show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

1/II/11H **Methods**

Use the substitution $y = x^p$ to find the general solution of

$$\mathcal{L}_x y \equiv \frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0.$$

Find the Green's function $G(x,\xi)$, $0 < \xi < \infty$, which satisfies

$$\mathcal{L}_x G(x,\xi) = \delta(x-\xi)$$

for $x > 0$, subject to the boundary conditions $G(x,\xi) \to 0$ as $x \to 0$ and as $x \to \infty$, for each fixed ξ.

Hence, find the solution of the equation

$$\mathcal{L}_x y = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & x > 1, \end{cases}$$

subject to the same boundary conditions.

Verify that both forms of your solution satisfy the appropriate equation and boundary conditions, and match at $x = 1$.

2/I/2G **Methods**

Show that the symmetric and antisymmetric parts of a second-rank tensor are themselves tensors, and that the decomposition of a tensor into symmetric and antisymmetric parts is unique.

For the tensor A having components

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix},$$

find the scalar a, vector $\mathbf{p}$ and symmetric traceless tensor B such that

$$A\mathbf{x} = a\mathbf{x} + \mathbf{p} \wedge \mathbf{x} + B\mathbf{x}$$

for every vector $\mathbf{x}$.

2/II/11G **Methods**

Explain what is meant by an *isotropic* tensor.

Show that the fourth-rank tensor

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \qquad (*)$$

is isotropic for arbitrary scalars α, β and γ.

Assuming that the most general isotropic tensor of rank 4 has the form $(*)$, or otherwise, evaluate

$$B_{ijkl} = \int_{r<a} x_i x_j \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV,$$

where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$.

3/I/2G Methods

Laplace's equation in the plane is given in terms of plane polar coordinates r and θ in the form

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

In each of the cases

$$\text{(i)} \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad \text{(ii)} \quad 1 \leqslant r < \infty,$$

find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \leqslant r \leqslant b$ with the boundary conditions

$$\phi = 1 \quad \text{on} \quad r = a \quad \text{for} \quad \text{all} \quad \theta,$$

$$\phi = 2 \quad \text{on} \quad r = b \quad \text{for} \quad \text{all} \quad \theta.$$

3/II/12H Methods

Find the Fourier sine series representation on the interval $0 \leqslant x \leqslant l$ of the function

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant b, \\ 0, & b < x \leqslant l. \end{cases}$$

The motion of a struck string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2}, \quad \text{for} \quad 0 \leqslant x \leqslant l \quad \text{and} \quad t \geqslant 0,$$

subject to boundary conditions $y = 0$ at $x = 0$ and $x = l$ for $t \geqslant 0$, and to the initial conditions $y = 0$ and $\dfrac{\partial y}{\partial t} = \delta(x - \frac{1}{4}l)$ at $t = 0$.

Obtain the solution $y(x,t)$ for this motion. Evaluate $y(x,t)$ for $t = \frac{1}{2}l/c$, and sketch it clearly.

4/I/2H Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0, 1, \ldots, \; -1 \leqslant x \leqslant 1.$$

Show that $R_n(x) = P_n'(x)$ obeys an equation which can be recast in Sturm–Liouville form and has the eigenvalue $(n-1)(n+2)$. What is the orthogonality relation for $R_n(x), R_m(x)$ for $n \neq m$?

4/II/11H **Methods**

A curve $y(x)$ in the xy-plane connects the points $(\pm a, 0)$ and has a fixed length l, $2a < l < \pi a$. Find an expression for the area A of the surface of the revolution obtained by rotating $y(x)$ about the x-axis.

Show that the area A has a stationary value for

$$y = \frac{1}{k}(\cosh kx - \cosh ka),$$

where k is a constant such that

$$lk = 2\sinh ka.$$

Show that the latter equation admits a unique positive solution for k.

1/I/9F **Quantum Mechanics**

A quantum mechanical particle of mass m and energy E encounters a potential step,

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geqslant 0. \end{cases}$$

Calculate the probability P that the particle is reflected in the case $E > V_0$.

If V_0 is positive, what is the limiting value of P when E tends to V_0? If V_0 is negative, what is the limiting value of P as V_0 tends to $-\infty$ for fixed E?

1/II/18F **Quantum Mechanics**

Consider a quantum-mechanical particle of mass m moving in a potential well,

$$V(x) = \begin{cases} 0, & -a < x < a, \\ \infty, & \text{elsewhere.} \end{cases}$$

(a) Verify that the set of normalised energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{1}{a}}\ \sin\left(\frac{n\pi(x+a)}{2a}\right), \qquad n = 1, 2, \ldots,$$

and evaluate the corresponding energy eigenvalues E_n.

(b) At time $t = 0$ the wavefunction for the particle is only nonzero in the positive half of the well,

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{elsewhere.} \end{cases}$$

Evaluate the expectation value of the energy, first using

$$\langle E \rangle = \int_{-a}^{a} \psi H \psi dx,$$

and secondly using

$$\langle E \rangle = \sum_n |a_n|^2 E_n,$$

where the a_n are the expansion coefficients in

$$\psi(x) = \sum_n a_n \psi_n(x).$$

Hence, show that

$$1 = \frac{1}{2} + \frac{8}{\pi^2} \sum_{p=0}^{\infty} \frac{(2p+1)^2}{[(2p+1)^2 - 4]^2}\ .$$

2/I/9F Quantum Mechanics

Consider a solution $\psi(x,t)$ of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$. The expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\ \mathcal{O}\ \rangle = \int\ dx\ \psi^*(x,t)\ \mathcal{O}\ \psi(x,t).$$

Show that

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m},$$

where

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

and that

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}(x)\right\rangle.$$

[*You may assume that* $\psi(x,t)$ *vanishes as* $x \to \pm\infty$.]

2/II/18F Quantum Mechanics

(a) Write down the angular momentum operators L_1, L_2, L_3 in terms of x_i and

$$p_i = -i\hbar\frac{\partial}{\partial x_i},\ i = 1,2,3.$$

Verify the commutation relation

$$[L_1, L_2] = i\hbar L_3.$$

Show that this result and its cyclic permutations imply

$$[L_3, L_1 \pm iL_2] = \pm\hbar\ (L_1 \pm iL_2),$$
$$[\mathbf{L}^2, L_1 \pm iL_2] = 0.$$

(b) Consider a wavefunction of the form $\psi = (x_3^2 + ar^2)f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that for a particular value of a, ψ is an eigenfunction of both $\mathbf{L}^2$ and L_3. What are the corresponding eigenvalues?

3/II/20F **Quantum Mechanics**

A quantum system has a complete set of orthonormalised energy eigenfunctions $\psi_n(x)$ with corresponding energy eigenvalues $E_n,\ n = 1, 2, 3, \ldots$.

(a) If the time-dependent wavefunction $\psi(x,t)$ is, at $t = 0$,

$$\psi(x,0) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

determine $\psi(x,t)$ for all $t > 0$.

(b) A linear operator $\mathcal{S}$ acts on the energy eigenfunctions as follows:

$$\begin{aligned} \mathcal{S}\psi_1 &= 7\psi_1 + 24\psi_2, \\ \mathcal{S}\psi_2 &= 24\psi_1 - 7\psi_2, \\ \mathcal{S}\psi_n &= 0, \quad n \geqslant 3. \end{aligned}$$

Find the eigenvalues of $\mathcal{S}$. At time $t = 0$, $\mathcal{S}$ is measured and its lowest eigenvalue is found. At time $t > 0$, $\mathcal{S}$ is measured again. Show that the probability for obtaining the lowest eigenvalue again is

$$\frac{1}{625}\left(337 + 288\cos(\omega t)\right),$$

where $\omega = (E_1 - E_2)/\hbar$.

3/I/10F Special Relativity

A particle of rest mass m and four-momentum $P = (E/c, \mathbf{p})$ is detected by an observer with four-velocity U, satisfying $U \cdot U = c^2$, where the product of two four-vectors $P = (p^0, \mathbf{p})$ and $Q = (q^0, \mathbf{q})$ is $P \cdot Q = p^0 q^0 - \mathbf{p} \cdot \mathbf{q}$.

Show that the speed of the detected particle in the observer's rest frame is

$$v = c\,\sqrt{1 - \frac{P \cdot P c^2}{(P \cdot U)^2}}.$$

4/I/9F Special Relativity

What is Einstein's principle of relativity?

Show that a spherical shell expanding at the speed of light, $\mathbf{x}^2 = c^2 t^2$, in one coordinate system $(t, \mathbf{x})$, is still spherical in a second coordinate system $(t', \mathbf{x}')$ defined by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{u}{c}x\right), \\ x' &= \gamma(x - ut), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$. Why do these equations define a Lorentz transformation?

4/II/18F Special Relativity

A particle of mass M is at rest at $x = 0$, in coordinates (t, x). At time $t = 0$ it decays into two particles A and B of equal mass $m < M/2$. Assume that particle A moves in the *negative* x direction.

(a) Using relativistic energy and momentum conservation compute the energy, momentum and velocity of both particles A and B.

(b) After a proper time τ, measured in its own rest frame, particle A decays. Show that the spacetime coordinates of this event are

$$\begin{aligned} t &= \frac{M}{2m}\tau, \\ x &= -\frac{MV}{2m}\tau, \end{aligned}$$

where $V = c\sqrt{1 - 4(m/M)^2}$.

1/I/6G **Fluid Dynamics**

Determine the pressure at a depth z below the surface of a static fluid of density ρ subject to gravity g. A rigid body having volume V is fully submerged in such a fluid. Calculate the buoyancy force on the body.

An iceberg of uniform density ρ_I is observed to float with volume V_I protruding above a large static expanse of seawater of density ρ_w. What is the total volume of the iceberg?

1/II/15G **Fluid Dynamics**

A fluid motion has velocity potential $\phi(x, y, t)$ given by

$$\phi = \epsilon y \cos(x - t)$$

where ϵ is a constant. Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\nabla \cdot \mathbf{u}$.

The *time-average* of a quantity $\psi(x, y, t)$ is defined as $\frac{1}{2\pi}\int_0^{2\pi} \psi(x, y, t)\, dt$.

Show that the time-average of this velocity field at every point (x, y) is zero.

Write down an expression for the fluid acceleration and find the time-average acceleration at (x, y).

Suppose now that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at time $t = 0$ is marked with dye. Write down equations for its subsequent motion and verify that its position (x, y) at time $t > 0$ is given (correct to terms of order ϵ^2) as

$$\begin{aligned} x &= \epsilon^2(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t), \\ y &= \epsilon \sin t\ . \end{aligned}$$

Deduce the time-average velocity of the dyed particle correct to this order.

3/I/8G **Fluid Dynamics**

Inviscid incompressible fluid occupies the region $y > 0$, which is bounded by a rigid barrier along $y = 0$. At time $t = 0$, a line vortex of strength κ is placed at position (a, b). By considering the flow due to an image vortex at $(a, -b)$, or otherwise, determine the velocity potential in the fluid.

Derive the position of the original vortex at time $t > 0$.

3/II/18G **Fluid Dynamics**

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

A circular cylinder of radius a is immersed in unbounded inviscid fluid of uniform density ρ. The cylinder moves in a prescribed direction perpendicular to its axis, with speed U. Use cylindrical polar coordinates, with the direction $\theta = 0$ parallel to the direction of the motion, to find the velocity potential in the fluid.

If U depends on time t and gravity is negligible, determine the pressure field in the fluid at time t. Deduce the fluid force per unit length on the cylinder.

[In cylindrical polar coordinates, $\nabla\phi = \dfrac{\partial\phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$.]

4/I/7G **Fluid Dynamics**

Starting from the Euler equation, derive the *vorticity equation* for the motion of an inviscid incompressible fluid under a conservative body force, and give a physical interpretation of each term in the equation. Deduce that in a flow field of the form $\mathbf{u} = (u(x,y,t), v(x,y,t), 0)$ the vorticity of a material particle is conserved.

Find the vorticity for such a flow in terms of the stream function ψ. Deduce that if the flow is steady, there must be a function f such that

$$\nabla^2\psi = f(\psi)\ .$$

4/II/16G **Fluid Dynamics**

A long straight canal has rectangular cross-section with a horizontal bottom and width $w(x)$ that varies slowly with distance x downstream. Far upstream, w has a constant value W, the water depth has a constant value H, and the velocity has a constant value U. Assuming that the water velocity is steady and uniform across the channel, use mass conservation and Bernoulli's theorem, which should be stated carefully, to show that the water depth $h(x)$ satisfies

$$\left(\frac{W}{w}\right)^2 = \left(1 + \frac{2}{F}\right)\left(\frac{h}{H}\right)^2 - \frac{2}{F}\left(\frac{h}{H}\right)^3 \quad \text{where} \quad F = \frac{U^2}{gH}\ .$$

Deduce that for a given value of F, a flow of this kind can exist only if $w(x)$ is everywhere greater than or equal to a critical value w_c, which is to be determined in terms of F.

Suppose that $w(x) > w_c$ everywhere. At locations where the channel width exceeds W, determine graphically, or otherwise, under what circumstances the water depth exceeds H.

2/I/5E Numerical Analysis

Find an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ -4 & 3 & -4 & -2 \\ 4 & -2 & 3 & 6 \\ -6 & 5 & -8 & 1 \end{pmatrix},$$

and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 11 \end{pmatrix}.$$

2/II/14E Numerical Analysis

(a) Let B be an $n \times n$ positive-definite, symmetric matrix. Define the Cholesky factorization of B and prove that it is unique.

(b) Let A be an $m \times n$ matrix, $m \geqslant n$, such that $\operatorname{rank} A = n$. Prove the uniqueness of the "skinny QR factorization"

$$A = QR,$$

where the matrix Q is $m \times n$ with orthonormal columns, while R is an $n \times n$ upper-triangular matrix with positive diagonal elements.

[*Hint: Show that you may choose R as a matrix that features in the Cholesky factorization of $B = A^T A$.*]

3/I/6E Numerical Analysis

Given $f \in C^{n+1}[a, b]$, let the nth-degree polynomial p interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n \in [a, b]$ are distinct. Given $x \in [a, b]$, find the error $f(x) - p(x)$ in terms of a derivative of f.

3/II/16E **Numerical Analysis**

Let the monic polynomials p_n, $n \geqslant 0$, be orthogonal with respect to the weight function $w(x) > 0$, $a < x < b$, where the degree of each p_n is exactly n.

(a) Prove that each p_n, $n \geqslant 1$, has n distinct zeros in the interval (a, b).

(b) Suppose that the p_n satisfy the three-term recurrence relation

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n^2 p_{n-2}(x), \quad n \geqslant 2,$$

where $p_0(x) \equiv 1$, $p_1(x) = x - a_1$. Set

$$A_n = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}, \quad n \geqslant 2.$$

Prove that $p_n(x) = \det(xI - A_n)$, $n \geqslant 2$, and deduce that all the eigenvalues of A_n reside in (a, b).

1/I/3D **Statistics**

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a two-dimensional sufficient statistic for μ, quoting carefully, without proof, any result you use.

What is the maximum likelihood estimator of μ?

1/II/12D **Statistics**

What is a *simple hypothesis*? Define the terms *size* and *power* for a test of one simple hypothesis against another.

State, without proof, the Neyman–Pearson lemma.

Let X be a **single** random variable, with distribution F. Consider testing the null hypothesis $H_0 : F$ is standard normal, $N(0,1)$, against the alternative hypothesis $H_1 : F$ is double exponential, with density $\frac{1}{4}e^{-|x|/2}, x \in \mathbb{R}$.

Find the test of size $\alpha, \alpha < \frac{1}{4}$, which maximises power, and show that the power is $e^{-t/2}$, where $\Phi(t) = 1 - \alpha/2$ and Φ is the distribution function of $N(0,1)$.

[*Hint: if* $X \sim N(0,1),\ P(|X| > 1) = 0.3174.$]

2/I/3D **Statistics**

Suppose the **single** random variable X has a uniform distribution on the interval $[0, \theta]$ and it is required to estimate θ with the loss function

$$L(\theta, a) = c(\theta - a)^2,$$

where $c > 0$.

Find the posterior distribution for θ and the optimal Bayes point estimate with respect to the prior distribution with density $p(\theta) = \theta e^{-\theta},\ \theta > 0$.

2/II/12D Statistics

What is meant by a *generalized likelihood ratio test*? Explain in detail how to perform such a test.

Let $X_1, \ldots, X_n$ be independent random variables, and let X_i have a Poisson distribution with unknown mean λ_i, $i = 1, \ldots, n$.

Find the form of the generalized likelihood ratio statistic for testing $H_0 : \lambda_1 = \ldots = \lambda_n$, and show that it may be approximated by

$$\frac{1}{\bar{X}} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

If, for $n = 7$, you found that the value of this statistic was 27.3, would you accept H_0? Justify your answer.

4/I/3D Statistics

Consider the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

$i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are given constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, with σ^2 unknown.

Find the least squares estimator $\widehat{\beta}$ of β. State, without proof, the distribution of $\widehat{\beta}$ and describe how you would test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$, where β_0 is given.

4/II/12D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, where μ and σ^2 are unknown.

Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2, based on $X_1, \ldots, X_n$. Show that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent, and derive their distributions.

Suppose now it is intended to construct a "prediction interval" $I(X_1, \ldots, X_n)$ for a future, independent, $N(\mu, \sigma^2)$ random variable X_0. We require

$$P\Big\{X_0 \in I(X_1, \ldots, X_n)\Big\} = 1 - \alpha,$$

with the probability over the *joint* distribution of $X_0, X_1, \ldots, X_n$.

Let

$$I_\gamma(X_1, \ldots, X_n) = \left(\widehat{\mu} - \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}},\ \widehat{\mu} + \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}}\right).$$

By considering the distribution of $(X_0 - \widehat{\mu})/(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}})$, find the value of γ for which $P\{X_0 \in I_\gamma(X_1, \ldots, X_n)\} = 1 - \alpha$.

3/I/5D **Optimization**

Let $a_1, \ldots, a_n$ be given constants, not all equal.

Use the Lagrangian sufficiency theorem, which you should state clearly, without proof, to minimize $\sum_{i=1}^{n} x_i^2$ subject to the two constraints $\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} a_i x_i = 0$.

3/II/15D **Optimization**

Consider the following linear programming problem,

$$\begin{array}{ll} \text{minimize} & (3-p)x_1 + px_2 \\ \text{subject to} & \begin{array}{rl} 2x_1 + x_2 & \geqslant 8 \\ x_1 + 3x_2 & \geqslant 9 \\ x_1 \quad\quad & \leqslant 6 \\ x_1,\ x_2 \quad & \geqslant 0. \end{array} \end{array}$$

Formulate the problem in a suitable way for solution by the two-phase simplex method.

Using the two-phase simplex method, show that if $2 \leqslant p \leqslant \frac{9}{4}$ then the optimal solution has objective function value $9-p$, while if $\frac{9}{4} < p \leqslant 3$ the optimal objective function value is $18-5p$.

4/I/5D **Optimization**

Explain what is meant by a two-person zero-sum game with payoff matrix $A = [a_{ij}]$. Write down a set of sufficient conditions for a pair of strategies to be optimal for such a game.

A fair coin is tossed and the result is shown to player I, who must then decide to 'pass' or 'bet'. If he passes, he must pay player II £1. If he bets, player II, who does not know the result of the coin toss, may either 'fold' or 'call the bet'. If player II folds, she pays player I £1. If she calls the bet and the toss was a head, she pays player I £2; if she calls the bet and the toss was a tail, player I must pay her £2.

Formulate this as a two-person zero-sum game and find optimal strategies for the two players. Show that the game has value $\frac{1}{3}$.

[*Hint: Player I has four possible moves and player II two.*]

4/II/14D **Optimization**

Dumbledore Publishers must decide how many copies of the best-selling "History of Hogwarts" to print in the next two months to meet demand. It is known that the demands will be for 40 thousand and 60 thousand copies in the first and second months respectively, and these demands must be met on time. At the beginning of the first month, a supply of 10 thousand copies is available, from existing stock. During each month, Dumbledore can produce up to 40 thousand copies, at a cost of 400 galleons per thousand copies. By having employees work overtime, up to 150 thousand additional copies can be printed each month, at a cost of 450 galleons per thousand copies. At the end of each month, after production and the current month's demand has been satisfied, a holding cost of 20 galleons per thousand copies is incurred.

Formulate a transportation problem, with 5 supply points and 3 demand points, to minimize the sum of production and holding costs during the two month period, and solve it.

[*You may assume that copies produced during a month can be used to meet demand in that month.*]

1/I/8B **Quadratic Mathematics**

Let $q(x,y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Define what is meant by the *discriminant* d of q, and show that q is positive-definite if and only if $a > 0 > d$. Define what it means for the form q to be *reduced.* For any integer $d < 0$, we define the class number $h(d)$ to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant d. Show that $h(d)$ is always finite (for negative d). Find $h(-39)$, and exhibit the corresponding reduced forms.

1/II/17B **Quadratic Mathematics**

Let ϕ be a symmetric bilinear form on a finite dimensional vector space V over a field k of characteristic $\neq 2$. Prove that the form ϕ may be diagonalized, and interpret the rank r of ϕ in terms of the resulting diagonal form.

For ϕ a symmetric bilinear form on a real vector space V of finite dimension n, define the *signature* σ of ϕ, proving that it is well-defined. A subspace U of V is called *null* if $\phi|_U \equiv 0$; show that V has a null subspace of dimension $n - \frac{1}{2}(r + |\sigma|)$, but no null subspace of higher dimension.

Consider now the quadratic form q on $\mathbb{R}^5$ given by

$$2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).$$

Write down the matrix A for the corresponding symmetric bilinear form, and calculate $\det A$. Hence, or otherwise, find the rank and signature of q.

2/I/8B **Quadratic Mathematics**

Let V be a finite-dimensional vector space over a field k. Describe a bijective correspondence between the set of bilinear forms on V, and the set of linear maps of V to its dual space V^*. If ϕ_1, ϕ_2 are non-degenerate bilinear forms on V, prove that there exists an isomorphism $\alpha : V \to V$ such that $\phi_2(u,v) = \phi_1(u, \alpha v)$ for all $u, v \in V$. If furthermore both ϕ_1, ϕ_2 are symmetric, show that α is self-adjoint (i.e. equals its adjoint) with respect to ϕ_1.

2/II/17B Quadratic Mathematics

Suppose p is an odd prime and a an integer coprime to p. Define the Legendre symbol $(\frac{a}{p})$, and state (without proof) Euler's criterion for its calculation.

For j any positive integer, we denote by r_j the (unique) integer with $|r_j| \leq (p-1)/2$ and $r_j \equiv aj \bmod p$. Let l be the number of integers $1 \leq j \leq (p-1)/2$ for which r_j is negative. Prove that

$$\left(\frac{a}{p}\right) = (-1)^l .$$

Hence determine the odd primes for which 2 is a quadratic residue.

Suppose that $p_1, \ldots, p_m$ are primes congruent to 7 modulo 8, and let

$$N \ = \ 8(p_1 \ldots p_m)^2 - 1 .$$

Show that 2 is a quadratic residue for any prime dividing N. Prove that N is divisible by some prime $p \equiv 7 \bmod 8$. Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

3/I/9B Quadratic Mathematics

Let A be the Hermitian matrix

$$\begin{pmatrix} 1 & i & 2i \\ -i & 3 & -i \\ -2i & i & 5 \end{pmatrix} .$$

Explaining carefully the method you use, find a diagonal matrix D with **rational** entries, and an invertible (complex) matrix T such that $T^* D T = A$, where T^* here denotes the conjugated transpose of T.

Explain briefly why we cannot find T, D as above with T unitary.

[*You may assume that if a monic polynomial* $t^3 + a_2 t^2 + a_1 t + a_0$ *with integer coefficients has all its roots rational, then all its roots are in fact integers.*]

3/II/19B **Quadratic Mathematics**

Let J_1 denote the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that T is a 2×2 upper-triangular real matrix with strictly positive diagonal entries and that ${J_1}^{-1} T J_1 T^{-1}$ is orthogonal. Verify that $J_1 T = T J_1$.

Prove that any real invertible matrix A has a decomposition $A = BC$, where B is an orthogonal matrix and C is an upper-triangular matrix with strictly positive diagonal entries.

Let A now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let K denote the $2n \times 2n$ matrix with n copies of J_1 on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1} C K C^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1} C K C^{-1}$ are zero, apart from n orthogonal 2×2 blocks $E_1, \ldots, E_n$ along the diagonal. Show that each E_i has the form ${J_1}^{-1} C_i J_1 {C_i}^{-1}$, for some 2×2 upper-triangular matrix C_i with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[*Hint: The invertible* $2n \times 2n$ *matrices* S *with* 2×2 *blocks* $S_1, \ldots, S_n$ *along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.*]

2/I/4B Further Analysis

Define the terms *connected* and *path connected* for a topological space. If a topological space X is path connected, prove that it is connected.

Consider the following subsets of $\mathbb{R}^2$:

$$I = \{(x,0) :\ 0 \leq x \leq 1\}, \quad A = \{(0,y) :\ \tfrac{1}{2} \leq y \leq 1\}, \text{ and}$$

$$J_n = \{(n^{-1},\ y) :\ 0 \leq y \leq 1\} \quad \text{for } n \geq 1.$$

Let

$$X = A \cup I \cup \bigcup_{n \geq 1} J_n$$

with the subspace (metric) topology. Prove that X is connected.

[*You may assume that any interval in* $\mathbb{R}$ *(with the usual topology) is connected.*]

2/II/13A Further Analysis

State Liouville's Theorem. Prove it by considering

$$\int_{|z|=R} \frac{f(z)\ dz}{(z-a)(z-b)}$$

and letting $R \to \infty$.

Prove that, if $g(z)$ is a function analytic on all of $\mathbb{C}$ with real and imaginary parts $u(z)$ and $v(z)$, then either of the conditions:

$$\text{(i) } u + v \geqslant 0 \text{ for all } z; \quad \text{or} \quad \text{(ii) } uv \geqslant 0 \text{ for all } z,$$

implies that $g(z)$ is constant.

3/I/3B Further Analysis

State a version of Rouché's Theorem. Find the number of solutions (counted with multiplicity) of the equation

$$z^4 = a(z-1)(z^2-1) + \tfrac{1}{2}$$

inside the open disc $\{z : |z| < \sqrt{2}\}$, for the cases $a = \frac{1}{3}, 12$ and 5.

[*Hint: For the case* $a = 5$*, you may find it helpful to consider the function* $(z^2-1)(z-2)(z-3)$.]

3/II/13B **Further Analysis**

If X and Y are topological spaces, describe the open sets in the *product topology* on $X \times Y$. If the topologies on X and Y are induced from metrics, prove that the same is true for the product.

What does it mean to say that a topological space is *compact*? If the topologies on X and Y are compact, prove that the same is true for the product.

4/I/4A **Further Analysis**

Let $f(z)$ be analytic in the disc $|z| < R$. Assume the formula

$$f'(z_0) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f(z)\, dz}{(z - z_0)^2}, \quad 0 \leqslant |z_0| < r < R.$$

By combining this formula with a complex conjugate version of Cauchy's Theorem, namely

$$0 = \int\limits_{|z|=r} \overline{f(z)}\, d\bar{z},$$

prove that

$$f'(0) = \frac{1}{\pi r} \int_0^{2\pi} u(\theta) e^{-i\theta} d\theta,$$

where $u(\theta)$ is the real part of $f(re^{i\theta})$.

4/II/13B **Further Analysis**

Let $\Delta^* = \{z : 0 < |z| < r\}$ be a punctured disc, and f an analytic function on Δ^*. What does it mean to say that f has the origin as (i) a removable singularity, (ii) a pole, and (iii) an essential singularity? State criteria for (i), (ii), (iii) to occur, in terms of the Laurent series for f at 0.

Suppose now that the origin is an essential singularity for f. Given any $w \in \mathbb{C}$, show that there exists a sequence (z_n) of points in Δ^* such that $z_n \to 0$ and $f(z_n) \to w$. [*You may assume the fact that an isolated singularity is removable if the function is bounded in some open neighbourhood of the singularity.*]

State the Open Mapping Theorem. Prove that if f is analytic and injective on Δ^*, then the origin cannot be an essential singularity. By applying this to the function $g(1/z)$, or otherwise, deduce that if g is an injective analytic function on $\mathbb{C}$, then g is linear of the form $az + b$, for some non-zero complex number a. [*Here, you may assume that g injective implies that its derivative g' is nowhere vanishing.*]