UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 4 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, ..., L*** *according to the letter affixed to each question. (For example,* ***1D, 13D*** *should be in one bundle and* ***8B, 9B*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

2H Principles of Dynamics

(i) Show that Newton's equations in Cartesian coordinates, for a system of N particles at positions $\mathbf{x}_i(t), i = 1, 2 \ldots N$, in a potential $V(\mathbf{x}, t)$, imply Lagrange's equations in a generalised coordinate system

$$q_j = q_j(\mathbf{x}_i, t) \quad , \quad j = 1, 2 \ldots 3N;$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad , \quad j = 1, 2 \ldots 3N,$$

where $L = T - V$, $T(q, \dot{q}, t)$ being the total kinetic energy and $V(q, t)$ the total potential energy.

(ii) Consider a light rod of length L, free to rotate in a vertical plane (the xz plane), but with one end P forced to move in the x-direction. The other end of the rod is attached to a heavy mass M upon which gravity acts in the negative z direction.

(a) Write down the Lagrangian for the system.

(b) Show that, if P is stationary, the rod has two equilibrium positions, one stable and the other unstable.

(c) The end at P is now forced to move with constant acceleration, $\ddot{x} = A$. Show that, once more, there is one stable equilibrium value of the angle the rod makes with the vertical, and find it.

3C Groups, Rings and Fields

(i) Define the notion of a Sylow p-subgroup of a finite group G, and state a theorem concerning the number of them and the relation between them.

(ii) Show that any group of order 30 has a non-trivial normal subgroup. Is it true that every group of order 30 is commutative?

4J Electromagnetism

(i) Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a current sheet, $\mathbf{J}$, with unit normal to the sheet $\mathbf{n}$, are

$$\mathbf{n} \wedge \mathbf{B}_2 - \mathbf{n} \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}.$$

State without proof the force per unit area on $\mathbf{J}$.

(ii) Conducting gas occupies the infinite slab $0 \leqslant x \leqslant a$. It carries a steady current $\mathbf{j} = (0, 0, j)$ and a magnetic field $\mathbf{B} = (0, B, 0)$ where $\mathbf{j}, \mathbf{B}$ depend only on x. The pressure is $p(x)$. The equation of hydrostatic equilibrium is $\nabla p = \mathbf{j} \wedge \mathbf{B}$. Write down the equations to be solved in this case. Show that $p + (1/2\mu_0)B^2$ is independent of x. Using the suffixes 1,2 to denote values at $x = 0, a$, respectively, verify that your results are in agreement with those of Part (i) in the case of $a \to 0$.

Suppose that

$$j(x) = \frac{\pi j_0}{2a} \sin\left(\frac{\pi x}{a}\right), \quad B_1 = 0, \quad p_2 = 0.$$

Find $B(x)$ everywhere in the slab.

5A Combinatorics

Let $\mathcal{A} \subset [n]^{(r)}$ where $r \leqslant n/2$. Prove that, if $\mathcal{A}$ is 1-intersecting, then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$. State an upper bound on $|\mathcal{A}|$ that is valid if $\mathcal{A}$ is t-intersecting and n is large compared to r and t.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be maximal 1-intersecting; that is, $\mathcal{B}$ is 1-intersecting but if $\mathcal{B} \subset \mathcal{C} \subset \mathcal{P}([n])$ and $\mathcal{B} \neq \mathcal{C}$ then $\mathcal{C}$ is not 1-intersecting. Show that $|\mathcal{B}| = 2^{n-1}$.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be 2-intersecting. Show that $|\mathcal{B}| \geqslant 2^{n-2}$ is possible. Can the inequality be strict?

6C Representation Theory

Compute the character table of A_5 (begin by listing the conjugacy classes and their orders).

[*It is not enough to write down the result; you must justify your answer.*]

7C Galois Theory

Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbf{Q}$ is of order 6. By explicitly describing the elements of G, show that they have orders $1, 2$ or 3. Hence deduce that G is isomorphic to S_3.

Why does it follow that $X^6 + 3$ is reducible over the finite field $\mathbf{F}_p$, for all primes p?

8B Differentiable Manifolds

Define an immersion and an embedding of one manifold in another. State a necessary and sufficient condition for an immersion to be an embedding and prove its necessity.

Assuming the existence of "bump functions" on Euclidean spaces, state and prove a version of Whitney's embedding theorem.

Deduce that $\mathbb{RP}^n$ embeds in $\mathbb{R}^{(n+1)^2}$.

9B Number Fields

Let $K = \mathbf{Q}(\alpha)$ be a number field, where $\alpha \in \mathcal{O}_K$. Let f be the (normalized) minimal polynomial of α over $\mathbf{Q}$. Show that the discriminant $\operatorname{disc}(f)$ of f is equal to $(\mathcal{O}_K : \mathbf{Z}[\alpha])^2 D_K$.

Show that $f(x) = x^3 + 5x^2 - 19$ is irreducible over $\mathbf{Q}$. Determine $\operatorname{disc}(f)$ and the ring of algebraic integers $\mathcal{O}_K$ of $K = \mathbf{Q}(\alpha)$, where $\alpha \in \mathbf{C}$ is a root of f.

10A Hilbert Spaces

State and prove the Riesz representation theorem for bounded linear functionals on a Hilbert space H.

[*You may assume, without proof, that* $H = E \oplus E^{\perp}$, *for every closed subspace* E *of* H.]

Prove that, for every $T \in \mathcal{B}(H)$, there is a unique $T^* \in \mathcal{B}(H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for every $x, y \in H$. Prove that $\|T^*T\| = \|T\|^2$ for every $T \in \mathcal{B}(H)$.

Define a *normal* operator $T \in \mathcal{B}(H)$. Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$. Deduce that every point in the spectrum of a normal operator T is an approximate eigenvalue of T.

[*You may assume, without proof, any general criterion for the invertibility of a bounded linear operator on* H.]

11B Riemann Surfaces

Recall that an *automorphism* of a Riemann surface is a bijective analytic map onto itself, and that the inverse map is then guaranteed to be analytic.

Let Δ denote the disc $\{z \in \mathbb{C} \,\big|\, |z| < 1\}$, and let $\Delta^* = \Delta - \{0\}$.

(a) Prove that an automorphism $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ is a Euclidian rotation.

[*Hint: Apply the maximum modulus principle to the functions* $\phi(z)/z$ *and* $\phi^{-1}(z)/z$.]

(b) Prove that a holomorphic map $\phi : \Delta^* \to \Delta$ extends to the entire disc, and use this to conclude that any automorphism of Δ^* is a Euclidean rotation.

[*You may use the result stated in part (a).*]

(c) Define an analytic map between Riemann surfaces. Show that a continuous map between Riemann surfaces, known to be analytic everywhere except perhaps at a single point P, is, in fact, analytic everywhere.

12B Logic, Computation and Set Theory

(i) What is the *Halting Problem*? What is an *unsolvable* problem?

(ii) Prove that the Halting Problem is unsolvable. Is it decidable whether or not a machine halts with input zero?

13D Probability and Measure

State and prove Hölder's Inequality.

[*Jensen's inequality, and other standard results, may be assumed.*]

Let (X_n) be a sequence of random variables bounded in L_p for some $p > 1$. Prove that (X_n) is uniformly integrable.

Suppose that $X \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $p \in (1, \infty)$. Show that $X \in L_r(\Omega, \mathcal{F}, \mathbb{P})$ for all $1 \leqslant r < p$ and that $||X||_r$ is an increasing function of r on $[1, p]$.

Show further that $\lim\limits_{r \to 1^+} ||X||_r = ||X||_1$.

14E Information Theory

Let $p_1, \ldots, p_n$ be a probability distribution, with $p^* = \max_i [p_i]$. Prove that

$$(i) - \sum_i p_i \log p_i \geqslant -p^* \log p^* - (1 - p^*) \log(1 - p^*);$$

$$(ii) - \sum_i p_i \log p_i \geqslant \log(1/p^*); \text{ and}$$

$$(iii) - \sum_i p_i \log p_i \geqslant 2(1 - p^*).$$

All logarithms are to base 2.

[*Hint: To prove (iii), it is convenient to use (i) for* $p^* \geqslant \frac{1}{2}$ *and (ii) for* $p^* \leqslant \frac{1}{2}$.]

Random variables X and Y with values x and y from finite 'alphabets' I and J represent the input and output of a transmission channel, with the conditional probability $p(x \mid y) = \mathbb{P}(X = x \mid Y = y)$. Let $h(p(\cdot \mid y))$ denote the entropy of the conditional distribution $p(\cdot \mid y)$, $y \in J$, and $h(X \mid Y)$ denote the conditional entropy of X given Y. Define the ideal observer decoding rule as a map $f : J \to I$ such that $p(f(y) \mid y) = \max_{x \in I} p(x \mid y)$ for all $y \in J$. Show that under this rule the error probability

$$\pi_{\text{er}}(y) = \sum_{\substack{x \in I \\ x \neq f(y)}} p(x \mid y)$$

satisfies $\pi_{\text{er}}(y) \leqslant \frac{1}{2} h(p(\cdot \mid y))$, and the expected value satisfies

$$\mathbb{E}\pi_{\text{er}}(Y) \leqslant \frac{1}{2} h(X \mid Y).$$

15E Principles of Statistics

(i) What are the main approaches by which prior distributions are specified in Bayesian inference?

Define the risk function of a decision rule d. Given a prior distribution, define what is meant by a Bayes decision rule and explain how this is obtained from the posterior distribution.

(ii) Dashing late into King's Cross, I discover that Harry must have already boarded the Hogwarts Express. I must therefore make my own way onto platform nine and three-quarters. Unusually, there are two guards on duty, and I will ask one of them for directions. It is safe to assume that one guard is a Wizard, who will certainly be able to direct me, and the other a Muggle, who will certainly not. But which is which? Before choosing one of them to ask for directions to platform nine and three-quarters, I have just enough time to ask one of them "Are you a Wizard?", and on the basis of their answer I must make my choice of which guard to ask for directions. I know that a Wizard will answer this question truthfully, but that a Muggle will, with probability $\frac{1}{3}$, answer it untruthfully.

Failure to catch the Hogwarts Express results in a loss which I measure as 1000 galleons, there being no loss associated with catching up with Harry on the train.

Write down an exhaustive set of non-randomised decision rules for my problem and, by drawing the associated risk set, determine my minimax decision rule.

My prior probability is $\frac{2}{3}$ that the guard I ask "Are you a Wizard?" is indeed a Wizard. What is my Bayes decision rule?

16D Stochastic Financial Models

(i) The price of the stock in the binomial model at time r, $1 \leqslant r \leqslant n$, is $S_r = S_0 \prod_{j=1}^{r} Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are independent, identically-distributed random variables with $\mathbb{P}(Y_1 = u) = p = 1 - \mathbb{P}(Y_1 = d)$ and the initial price S_0 is a constant. Denote the fixed interest rate on the bank account by ρ, where $u > 1 + \rho > d > 0$, and let the discount factor $\alpha = 1/(1+\rho)$. Determine the unique value $p = \overline{p}$ for which the sequence $\{\alpha^r S_r,\, 0 \leqslant r \leqslant n\}$ is a martingale.

Explain briefly the significance of $\overline{p}$ for the pricing of contingent claims in the model.

(ii) Let T_a denote the first time that a standard Brownian motion reaches the level $a > 0$. Prove that for $t > 0$,

$$\mathbb{P}(T_a \leqslant t) = 2\left[1 - \Phi\left(a/\sqrt{t}\right)\right],$$

where Φ is the standard normal distribution function.

Suppose that A_t and B_t represent the prices at time t of two different stocks with initial prices 1 and 2, respectively; the prices evolve so that they may be represented as $A_t = e^{\sigma_1 X_t + \mu t}$ and $B_t = 2e^{\sigma_2 Y_t + \mu t}$, respectively, where $\{X_t\}_{t \geqslant 0}$ and $\{Y_t\}_{t \geqslant 0}$ are independent standard Brownian motions and σ_1, σ_2 and μ are constants. Let T denote the first time, if ever, that the prices of the two stocks are the same. Determine $\mathbb{P}(T \leqslant t)$, for $t > 0$.

17K Dynamical Systems

Define topological conjugacy and C^1-conjugacy.

Let a, b be real numbers with $a > b > 0$ and let F_a, F_b be the maps of $(0, \infty)$ to itself given by $F_a(x) = ax, F_b(x) = bx$. For which pairs a, b are F_a and F_b topologically conjugate? Would the answer be the same for C^1-conjugacy? Justify your statements.

18A Partial Differential Equations

(a) Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

together with the boundary condition on the x-axis:

$$u(x,0) = f(x) \ ,$$

where f is a smooth function. You should discuss the domain on which the solution is smooth. For which functions f can the solution be extended to give a smooth solution on the upper half plane $\{y > 0\}$?

(b) Solve the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

together with the boundary condition on the unit circle:

$$u(x,y) = x \qquad \text{when} \qquad x^2 + y^2 = 1.$$

19L Methods of Mathematical Physics

State and prove the convolution theorem for Laplace transforms.

Use the convolution theorem to prove that the Beta function

$$B(p,q) = \int_0^1 (1-\tau)^{p-1}\tau^{q-1}d\tau$$

may be written in terms of the Gamma function as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ .$$

20K Numerical Analysis

(i) Let A be a symmetric $n \times n$ matrix such that

$$A_{k,k} > \sum_{\substack{l=1 \\ l \neq k}}^{n} |A_{k,l}| \qquad 1 \leqslant k \leqslant n.$$

Prove that it is positive definite.

(ii) Prove that both Jacobi and Gauss-Seidel methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, where the matrix A obeys the conditions of (i), converge.

[*You may quote the Householder-John theorem without proof.*]

21F Electrodynamics

Explain the multipole expansion in electrostatics, and devise formulae for the total charge, dipole moments and quadrupole moments given by a static charge distribution $\rho(\mathbf{r})$.

A nucleus is modelled as a uniform distribution of charge inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

The total charge of the nucleus is Q. What are the dipole moments and quadrupole moments of this distribution?

Describe qualitatively what happens if the nucleus starts to oscillate.

22F Statistical Physics

Write down the first law of thermodynamics in differential form for an infinitesimal reversible change in terms of the increments dE, dS and dV, where E, S and V are to be defined. Briefly give an interpretation of each term and deduce that

$$P = -\left(\frac{\partial E}{\partial V}\right)_S , \qquad T = \left(\frac{\partial E}{\partial S}\right)_V .$$

Define the specific heat at constant volume C_V and show that for an adiabatic change

$$C_V \, dT + \left(\left(\frac{\partial E}{\partial V}\right)_T + P\right) dV = 0 .$$

Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ,$$

where T is temperature and hence show that

$$\left(\frac{\partial E}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V .$$

An imperfect gas of volume V obeys the van der Waals equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT ,$$

where a and b are non-negative constants. Show that

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 ,$$

and deduce that C_V is a function of T only. It can further be shown that in this case C_V is independent of T. Hence show that

$$T(V - b)^{R/C_V}$$

is constant on adiabatic curves.

23J Applications of Quantum Mechanics

A steady beam of particles, having wavenumber k and moving in the z direction, scatters on a spherically-symmetric potential. Write down the asymptotic form of the wave function at large r.

The incoming wave is written as a partial-wave series

$$\sum_{\ell=0}^{\infty} \chi_\ell(kr) P_\ell(\cos\theta).$$

Show that for large r

$$\chi_\ell(kr) \sim \frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

and calculate $\chi_0(kr)$ and $\chi_1(kr)$ for all r.

Write down the second-order differential equation satisfied by the $\chi_\ell(kr)$. Construct a second linearly-independent solution for each ℓ that is singular at $r=0$ and, when it is suitably normalised, has large-r behaviour

$$\frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} + (-1)^\ell e^{-ikr}\right).$$

24J General Relativity

(i) The metric of any two-dimensional curved space, rotationally symmetric about a point P, can by suitable choice of coordinates be written locally in the form

$$ds^2 = e^{2\phi(r)}(dr^2 + r^2 d\theta^2),$$

where $r = 0$ at P, $r > 0$ away from P, and $0 \leqslant \theta < 2\pi$. Labelling the coordinates as $(x^1, x^2) = (r, \theta)$, show that the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{11}$ and Γ^2_{22} are each zero, and compute the non-zero Christoffel symbols $\Gamma^1_{11}, \Gamma^1_{22}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

The Ricci tensor R_{ab} $(a, b = 1, 2)$ is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd},$$

where a comma denotes a partial derivative. Show that $R_{12} = 0$ and that

$$R_{11} = -\phi'' - r^{-1}\phi', \quad R_{22} = r^2 R_{11}.$$

(ii) Suppose further that, in a neighbourhood of P, the Ricci scalar R takes the constant value -2. Find a second order differential equation, which you should denote by $(*)$, for $\phi(r)$.

This space of constant Ricci scalar can, by a suitable coordinate transformation $r \to \chi(r)$, leaving θ invariant, be written locally as

$$ds^2 = d\chi^2 + \sinh^2 \chi d\theta^2$$

By studying this coordinate transformation, or otherwise, find $\cosh \chi$ and $\sinh \chi$ in terms of r (up to a constant of integration). Deduce that

$$e^{\phi(r)} = \frac{2A}{(1 - A^2 r^2)} \quad , \quad (0 \leqslant Ar < 1),$$

where A is a positive constant and verify that your equation $(*)$ for ϕ holds.

[*Note that*

$$\int \frac{d\chi}{\sinh \chi} = \text{const.} + \frac{1}{2}\log\bigl(\cosh \chi - 1\bigr) - \frac{1}{2}\log\bigl(\cosh \chi + 1\bigr).\Bigr]$$

25H Fluid Dynamics II

The energy equation for the motion of a viscous, incompressible fluid states that

$$\frac{d}{dt}\int_{V(t)} \frac{1}{2}\rho u^2 dV = \int_{S(t)} u_i \sigma_{ij} n_j dS - 2\mu \int_{V(t)} e_{ij} e_{ij} dV.$$

Interpret each term in this equation and explain the meaning of the symbols used.

For steady rectilinear flow in a (not necessarily circular) pipe having rigid stationary walls, deduce a relation between the viscous dissipation per unit length of the pipe, the pressure gradient G, and the volume flux Q.

Starting from the Navier-Stokes equations, calculate the velocity field for steady rectilinear flow in a circular pipe of radius a. Using the relationship derived above, or otherwise, find in terms of G the viscous dissipation per unit length for this flow.

[*In cylindrical polar coordinates,*

$$\nabla^2 w(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) .]$$

26L Waves in Fluid and Solid Media

Derive Riemann's equations for finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

At time $t = 0$ the gas is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x = 0$ starts moving backwards at time $t = 0$ with displacement $x = -a\sin\omega t$, where a and ω are positive constants. Explain briefly how to find the resulting disturbance using a graphical construction in the xt-plane, and show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma - 1)u$.

For small amplitude a, show that the excess pressure $\Delta p = p - p_0$ and the excess sound speed $\Delta c = c - c_0$ are related by

$$\frac{\Delta p}{p_0} = \frac{2\gamma}{\gamma - 1}\frac{\Delta c}{c_0} + \frac{\gamma(\gamma+1)}{(\gamma-1)^2}\left(\frac{\Delta c}{c_0}\right)^2 + O\left(\left(\frac{\Delta c}{c_0}\right)^3\right).$$

Deduce that the time-averaged pressure on the face of the piston exceeds p_0 by

$$\frac{1}{8}\rho_0 a^2\omega^2(\gamma+1) + O(a^3).$$