UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part II Alternative A

Tuesday 5 June 2001 9.00 to 12.00

## PAPER 2

***Before you begin read these instructions carefully.***

*Each question is divided into Part **(i)** and Part **(ii)**, which may or may not be related. Candidates may attempt either or both Parts of any question, **but** must not attempt Parts from more than **SIX** questions.*

*The number of marks for each question is the same, with Part **(ii)** of each question carrying twice as many marks as Part **(i)**. Additional credit will be given for a substantially complete answer to either part.*

*Begin each answer on a separate sheet.*

*Write legibly and on only **one** side of the paper.*

***At the end of the examination:***

*Tie your answers in separate bundles, marked **A, B, C, ..., L** according to the letter affixed to each question. (For example, **8A, 9A** should be in one bundle and **10E, 12E** in another bundle.)*

*Attach a completed cover sheet to each bundle.*

*Complete a master cover sheet listing **all** Parts of **all** questions attempted.*

***It is essential that every cover sheet bear the candidate's examination number and desk number.***

**1D Markov Chains**

**(i)** The fire alarm in Mill Lane is set off at random times. The probability of an alarm during the time-interval $(u, u+h)$ is $\lambda(u)h + o(h)$ where the 'intensity function' $\lambda(u)$ may vary with the time $u$. Let $N(t)$ be the number of alarms by time $t$, and set $N(0) = 0$. Show, subject to reasonable extra assumptions to be stated clearly, that $p_i(t) = P(N(t) = i)$ satisfies

$$p_0'(t) = -\lambda(t)p_0(t), \quad p_i'(t) = \lambda(t)\{p_{i-1}(t) - p_i(t)\}, \quad i \geqslant 1.$$

Deduce that $N(t)$ has the Poisson distribution with parameter $\Lambda(t) = \int_0^t \lambda(u)du$.

**(ii)** The fire alarm in Clarkson Road is different. The number $M(t)$ of alarms by time $t$ is such that

$$P(M(t+h) = m+1 \mid M(t) = m) = \lambda_m h + o(h) \ ,$$

where $\lambda_m = \alpha m + \beta$, $m \geqslant 0$, and $\alpha, \beta > 0$. Show, subject to suitable extra conditions, that $p_m(t) = P(M(t) = m)$ satisfies a set of differential-difference equations to be specified. Deduce without solving these equations in their entirety that $M(t)$ has mean $\beta(e^{\alpha t} - 1)/\alpha$, and find the variance of $M(t)$.

**2H Principles of Dynamics**

**(i)** An axially symmetric top rotates freely about a fixed point $O$ on its axis. The principal moments of inertia are $A$, $A$, $C$ and the centre of gravity $G$ is a distance $h$ from $O$.

Define the three Euler angles $\theta$, $\phi$ and $\psi$, specifying the orientation of the top. Use Lagrange's equations to show that there are three conserved quantities in the motion. Interpret them physically.

**(ii)** Initially the top is spinning with angular speed $n$ about $OG$, with $OG$ vertical, before it is slightly disturbed.

Show that, in the subsequent motion, $\theta$ stays close to zero if $C^2n^2 > 4mghA$, but if this condition fails then $\theta$ attains a maximum value given approximately by

$$\cos\theta \approx \frac{C^2n^2}{2mghA} - 1.$$

Why is this only an approximation?

**3A Functional Analysis**

**(i)** State the Stone-Weierstrass theorem for complex-valued functions. Use it to show that the trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous, complex-valued functions on the unit circle $\mathbb{T}$ with the uniform norm.

Show further that, for $f \in C(\mathbb{T})$, the $n$th Fourier coefficient

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

tends to 0 as $|n|$ tends to infinity.

**(ii)** **(a)** Let $X$ be a normed space with the property that the series $\sum_{n=1}^{\infty} x_n$ converges whenever $(x_n)$ is a sequence in $X$ with $\sum_{n=1}^{\infty} ||x_n||$ convergent. Show that $X$ is a Banach space.

**(b)** Let $K$ be a compact metric space and $L$ a closed subset of $K$. Let $R : C(K) \to C(L)$ be the map sending $f \in C(K)$ to its restriction $R(f) = f|L$ to $L$. Show that $R$ is a bounded, linear map and that its image is a subalgebra of $C(L)$ separating the points of $L$.

Show further that, for each function $g$ in the image of $R$, there is a function $f \in C(K)$ with $R(f) = g$ and $||f||_{\infty} = ||g||_{\infty}$. Deduce that every continuous, complex-valued function on $L$ can be extended to a continuous function on all of $K$.

**4C Groups, Rings and Fields**

**(i)** Show that the ring $k = \mathbf{F}_2[X]/(X^2 + X + 1)$ is a field. How many elements does it have?

**(ii)** Let $k$ be as in (i). By considering what happens to a chosen basis of the vector space $k^2$, or otherwise, find the order of the groups $GL_2(k)$ and $SL_2(k)$.

By considering the set of lines in $k^2$, or otherwise, show that $SL_2(k)$ is a subgroup of the symmetric group $S_5$, and identify this subgroup.

**5J Electromagnetism**

**(i)** Write down the expression for the electrostatic potential $\phi(\mathbf{r})$ due to a distribution of charge $\rho(\mathbf{r})$ contained in a volume $V$. Perform the multipole expansion of $\phi(\mathbf{r})$ taken only as far as the dipole term.

**(ii)** If the volume $V$ is the sphere $|\mathbf{r}| \leqslant a$ and the charge distribution is given by

$$\rho(\mathbf{r}) = \begin{cases} r^2 \cos\theta & r \leqslant a \\ 0 & r > a \end{cases},$$

where $r, \theta$ are spherical polar coordinates, calculate the charge and dipole moment. Hence deduce $\phi$ as far as the dipole term.

Obtain an exact solution for $r > a$ by solving the boundary value problem using trial solutions of the forms

$$\phi = \frac{A\cos\theta}{r^2} \quad \text{for} \quad r > a,$$

and

$$\phi = Br\cos\theta + Cr^4\cos\theta \quad \text{for} \quad r < a.$$

Show that the solution obtained from the multipole expansion is in fact exact for $r > a$.

[*You may use without proof the result*

$$\nabla^2(r^k \cos\theta) = (k+2)(k-1)r^{k-2}\cos\theta, \qquad k \in \mathbb{N}.]$$

**6K Dynamics of Differential Equations**

**(i)** Define a Liapounov function for a flow $\phi$ on $\mathbb{R}^n$. Explain what it means for a fixed point of the flow to be Liapounov stable. State and prove Liapounov's first stability theorem.

**(ii)** Consider the damped pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0,$$

where $k > 0$. Show that there are just two fixed points (considering the phase space as an infinite cylinder), and that one of these is the origin and is Liapounov stable. Show further that the origin is asymptotically stable, and that the the $\omega$-limit set of each point in the phase space is one or other of the two fixed points, justifying your answer carefully.

[*You should state carefully any theorems you use in your answer, but you need not prove them.*]

**7C Geometry of Surfaces**

**(i)** Give the definition of the curvature $\kappa(t)$ of a plane curve $\gamma : [a,b] \longrightarrow \mathbf{R}^2$. Show that, if $\gamma : [a,b] \longrightarrow \mathbf{R}^2$ is a simple closed curve, then

$$\int_a^b \kappa(t)\, \|\dot{\gamma}(t)\|\, dt = 2\pi.$$

**(ii)** Give the definition of a geodesic on a parametrized surface in $\mathbf{R}^3$. Derive the differential equations characterizing geodesics. Show that a great circle on the unit sphere is a geodesic.

**8A Graph Theory**

**(i)** Prove that any graph $G$ drawn on a compact surface $S$ with negative Euler characteristic $E(S)$ has a vertex colouring that uses at most

$$h = \lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24E(S)}) \rfloor$$

colours.

Briefly discuss whether the result is still true when $E(S) \geqslant 0$.

**(ii)** Prove that a graph $G$ is $k$ edge-connected if and only if the removal of no set of less than $k$ edges from $G$ disconnects $G$.

[*If you use any form of Menger's theorem, you must prove it.*]

Let $G$ be a minimal example of a graph that requires $k+1$ colours for a vertex colouring. Show that $G$ must be $k$ edge-connected.

**9A Coding and Cryptography**

**(i)** Give brief answers to the following questions.

(a) What is a stream cypher?

(b) Explain briefly why a one-time pad is safe if used only once but becomes unsafe if used many times.

(c) What is a feedback register of length $d$? What is a linear feedback register of length $d$?

(d) A cypher stream is given by a linear feedback register of known length $d$. Show that, given plain text and cyphered text of length $2d$, we can find the complete cypher stream.

(e) State and prove a similar result for a general feedback register.

**(ii)** Describe the construction of a Reed-Muller code. Establish its information rate and its weight.

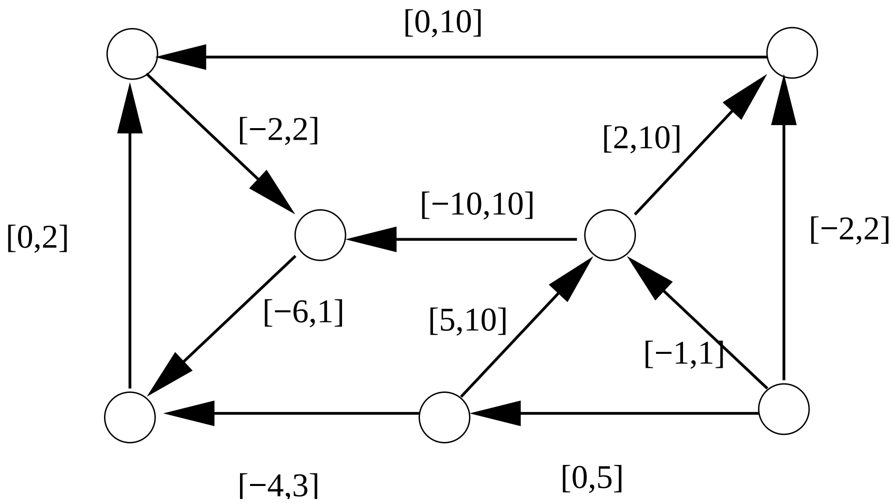
**10E Algorithms and Networks**

**(i)** Let $G$ be a directed network with nodes $N$ and arcs $A$. Let $S \subset N$ be a subset of the nodes, $x$ be a flow on $G$, and $y$ be the divergence of $x$. Describe carefully what is meant by a *cut* $Q = [S, N \backslash S]$. Define the *arc-cut incidence* $e_Q$, and the *flux of* $x$ *across* $Q$. Define also the *divergence* $y(S)$ *of* $S$. Show that $y(S) = x.e_Q$.

Now suppose that capacity constraints are specified on each of the arcs. Define the *upper cut capacity* $c^+(Q)$ *of* $Q$. State the feasible distribution problem for a specified divergence $b$, and show that the problem only has a solution if $b(N) = 0$ and $b(S) \leqslant c^+(Q)$ for all cuts $Q = [S, N \backslash S]$.

**(ii)** Describe an algorithm to find a feasible distribution given a specified divergence $b$ and capacity constraints on each arc. Explain what happens when no feasible distribution exists.

Illustrate the algorithm by either finding a feasible circulation, or demonstrating that one does not exist, in the network given below. Arcs are labelled with capacity constraint intervals.



**11E Principles of Statistics**

**(i)** Let $X_1, \ldots, X_n$ be independent, identically-distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a minimal sufficient statistic for $\mu$.

Let $T_1 = n^{-1} \sum_{i=1}^n X_i$ and $T_2 = \sqrt{n^{-1} \sum_{i=1}^n X_i^2}$. Write down the distribution of $X_i/\mu$, and hence show that $Z = T_1/T_2$ is ancillary. Explain briefly why the Conditionality Principle would lead to inference about $\mu$ being drawn from the conditional distribution of $T_2$ given $Z$.

What is the maximum likelihood estimator of $\mu$?

**(ii)** Describe briefly the Bayesian approach to predictive inference.

Let $Z_1, \ldots, Z_n$ be independent, identically-distributed $N(\mu, \sigma^2)$ random variables, with $\mu, \sigma^2$ both unknown. Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of $\mu, \sigma^2$ based on $Z_1, \ldots, Z_n$, and state, without proof, their joint distribution.

Suppose that it is required to construct a prediction interval $I_{1-\alpha} \equiv I_{1-\alpha}(Z_1, \ldots, Z_n)$ for a future, independent, random variable $Z_0$ with the same $N(\mu, \sigma^2)$ distribution, such that

$$P(Z_0 \in I_{1-\alpha}) = 1 - \alpha,$$

with the probability over the *joint* distribution of $Z_0, Z_1, \ldots, Z_n$. Let

$$I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2) = \left[\bar{Z}_n - z_{\alpha/2}\sigma\sqrt{1 + 1/n},\ \bar{Z}_n + z_{\alpha/2}\sigma\sqrt{1 + 1/n}\right],$$

where $\bar{Z}_n = n^{-1} \sum_{i=1}^n Z_i$, and $\Phi(z_\beta) = 1 - \beta$, with $\Phi$ the distribution function of $N(0, 1)$.

Show that $P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2)) = 1 - \alpha$.

By considering the distribution of $(Z_0 - \bar{Z}_n)/\left(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}}\right)$, or otherwise, show that

$$P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) < 1 - \alpha,$$

and show how to construct an interval $I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)$ with

$$P(Z_0 \in I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) = 1 - \alpha.$$

[*Hint: if $Y$ has the t-distribution with $m$ degrees of freedom and $t_\beta^{(m)}$ is defined by $P(Y < t_\beta^{(m)}) = 1 - \beta$ then $t_\beta > z_\beta$ for $\beta < \frac{1}{2}$.*]

**12E Computational Statistics and Statistical Modelling**

**(i)** Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that $Y_i$ has probability density function

$$f(y_i|\theta_i, \phi) = \exp[(y_i\theta_i - b(\theta_i))/\phi + c(y_i, \phi)].$$

Assume that $E(Y_i) = \mu_i$, and that $g(\mu_i) = \beta^T x_i$, where $g(\cdot)$ is a known 'link' function, $x_1, \ldots, x_n$ are known covariates, and $\beta$ is an unknown vector. Show that

$$\mathbb{E}(Y_i) = b'(\theta_i),\ \operatorname{var}(Y_i) = \phi b''(\theta_i) = V_i,\ \text{say},$$

and hence

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)x_i}{g'(\mu_i)V_i},\ \text{where } l = l(\beta, \phi) \text{ is the log-likelihood.}$$

**(ii)** The table below shows the number of train miles (in millions) and the number of collisions involving British Rail passenger trains between 1970 and 1984. Give a detailed interpretation of the $R$ output that is shown under this table:

| | year | collisions | miles |
|---|---|---|---|
| 1 | 1970 | 3 | 281 |
| 2 | 1971 | 6 | 276 |
| 3 | 1972 | 4 | 268 |
| 4 | 1973 | 7 | 269 |
| 5 | 1974 | 6 | 281 |
| 6 | 1975 | 2 | 271 |
| 7 | 1976 | 2 | 265 |
| 8 | 1977 | 4 | 264 |
| 9 | 1978 | 1 | 267 |
| 10 | 1979 | 7 | 265 |
| 11 | 1980 | 3 | 267 |
| 12 | 1981 | 5 | 260 |
| 13 | 1982 | 6 | 231 |
| 14 | 1983 | 1 | 249 |

```
Call:

glm(formula = collisions ~ year + log(miles), family = poisson)

Coefficients:
                   Estimate  Std. Error  z value  Pr(>|z|)
     (Intercept)  127.14453   121.37796    1.048     0.295
     year          -0.05398     0.05175   -1.043     0.297
     log(miles)    -3.41654     4.18616   -0.816     0.414

(Dispersion parameter for poisson family taken to be 1)

     Null deviance:  15.937 on 13 degrees of freedom

Residual deviance:  14.843 on 11 degrees of freedom

Number of Fisher Scoring iterations:  4
```

**13F Foundations of Quantum Mechanics**

**(i)** Hermitian operators $\hat{x}$, $\hat{p}$, satisfy $[\hat{x}, \hat{p}] = i\hbar$. The eigenvectors $|p\rangle$, satisfy $\hat{p}|p\rangle = p|p\rangle$ and $\langle p'|p\rangle = \delta(p' - p)$. By differentiating with respect to $b$ verify that

$$e^{-ib\hat{x}/\hbar}\hat{p}\, e^{ib\hat{x}/\hbar} = \hat{p} + b$$

and hence show that

$$e^{ib\hat{x}/\hbar}|p\rangle = |p + b\rangle.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

and

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle\,.$$

**(ii)** A quantum system has Hamiltonian $H = H_0 + H_1$, where $H_1$ is a small perturbation. The eigenvalues of $H_0$ are $\epsilon_n$. Give (without derivation) the formulae for the first order and second order perturbations in the energy level of a non-degenerate state. Suppose that the $r$th energy level of $H_0$ has $j$ degenerate states. Explain how to determine the eigenvalues of $H$ corresponding to these states to first order in $H_1$.

In a particular quantum system an orthonormal basis of states is given by $|n_1, n_2\rangle$, where $n_i$ are integers. The Hamiltonian is given by

$$H = \sum_{n_1,n_2}(n_1^2 + n_2^2)\,|n_1, n_2\rangle\langle n_1, n_2| + \sum_{n_1,n_2,n'_1,n'_2}\lambda_{|n_1 - n'_1|,|n_2 - n'_2|}\,|n_1, n_2\rangle\langle n'_1, n'_2|\,,$$

where $\lambda_{r,s} = \lambda_{s,r}$, $\lambda_{0,0} = 0$ and $\lambda_{r,s} = 0$ unless $r$ and $s$ are both even.

Obtain an expression for the ground state energy to second order in the perturbation, $\lambda_{r,s}$. Find the energy eigenvalues of the first excited state to first order in the perturbation. Determine a matrix (which depends on two independent parameters) whose eigenvalues give the first order energy shift of the second excited state.

**14F Quantum Physics**

**(i)** Each particle in a system of $N$ identical fermions has a set of energy levels, $E_i$, with degeneracy $g_i$, where $1 \leq i < \infty$. Explain why, in thermal equilibrium, the average number of particles with energy $E_i$ is

$$N_i = \frac{g_i}{e^{\beta(E_i-\mu)}+1}.$$

The physical significance of the parameters $\beta$ and $\mu$ should be made clear.

**(ii)** A simple model of a crystal consists of a linear array of sites with separation $a$. At the $n$th site an electron may occupy either of two states with probability amplitudes $b_n$ and $c_n$, respectively. The time-dependent Schrödinger equation governing the amplitudes gives

$$i\hbar\dot{b}_n = E_0 b_n - A(b_{n+1}+b_{n-1}+c_{n+1}+c_{n-1}),$$
$$i\hbar\dot{c}_n = E_1 c_n - A(b_{n+1}+b_{n-1}+c_{n+1}+c_{n-1}),$$

where $A>0$.

By examining solutions of the form

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} e^{i(kna-Et/\hbar)},$$

show that the energies of the electron fall into two bands given by

$$E = \frac{1}{2}(E_0+E_1-4A\cos ka) \pm \frac{1}{2}\sqrt{(E_0-E_1)^2+16A^2\cos^2 ka}.$$

Describe briefly how the energy band structure for electrons in real crystalline materials can be used to explain why they are insulators, conductors or semiconductors.

**15J General Relativity**

**(i)** Show that the geodesic equation follows from a variational principle with Lagrangian

$$L \;=\; g_{ab}\dot{x}^a\dot{x}^b$$

where the path of the particle is $x^a(\lambda)$, and $\lambda$ is an affine parameter along that path.

**(ii)** The Schwarzschild metric is given by

$$ds^2 = dr^2\left(1-\frac{2M}{r}\right)^{-1} + r^2(d\theta^2+\sin^2\theta d\phi^2) - \left(1-\frac{2M}{r}\right)dt^2.$$

Consider a photon which moves within the equatorial plane $\theta = \frac{\pi}{2}$. Using the above Lagrangian, or otherwise, show that

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E, \quad \text{and} \quad r^2\left(\frac{d\phi}{d\lambda}\right) = h,$$

for constants $E$ and $h$. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1-\frac{2M}{r}\right). \qquad (*)$$

Assume further that the photon approaches from infinity. Show that the impact parameter $b$ is given by

$$b = \frac{h}{E} \quad .$$

By considering the equation $(*)$, or otherwise

(a) show that, if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) show that, if $b^2 < 27M^2$, the photon is captured;

(c) describe, with justification, the qualitative form of the photon's orbit in the case $b^2 = 27M^2$.

**16L Theoretical Geophysics**

**(i)** In a reference frame rotating with constant angular velocity $\boldsymbol{\Omega}$ the equations of motion for an inviscid, incompressible fluid of density $\rho$ in a gravitational field $\mathbf{g} = -\nabla\Phi$ are

$$\rho\frac{D\mathbf{u}}{Dt} + 2\rho\boldsymbol{\Omega}\wedge\mathbf{u} = -\nabla p + \rho\mathbf{g}\ , \qquad \nabla\cdot\mathbf{u} = 0\ .$$

Define the Rossby number and explain what is meant by geostrophic flow.

Derive the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\mathbf{u} + \frac{\nabla\rho\wedge\nabla p}{\rho^2}.$$

[*Recall that* $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u}\wedge(\nabla\wedge\mathbf{u})$.]

Give a physical interpretation for the term $(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\mathbf{u}$.

**(ii)** Consider the rotating fluid of part (i), but now let $\rho$ be constant and absorb the effects of gravity into a modified pressure $P = p - \rho\mathbf{g}\cdot\mathbf{x}$. State the *linearized* equations of motion and the *linearized* vorticity equation for small-amplitude motions (inertial waves).

Use the linearized equations of motion to show that

$$\nabla^2 P = 2\rho\boldsymbol{\Omega}\cdot\boldsymbol{\omega}\ .$$

Calculate the time derivative of the curl of the linearized vorticity equation. Hence show that

$$\frac{\partial^2}{\partial t^2}(\nabla^2\mathbf{u}) = -(2\boldsymbol{\Omega}\cdot\nabla)^2\mathbf{u}\ .$$

Deduce the dispersion relation for waves proportional to $\exp[i(\mathbf{k}\cdot\mathbf{x} - nt)]$. Show that $|n| \leq 2\Omega$. Show further that if $n = 2\Omega$ then $P = 0$.

**17H Mathematical Methods**

**(i)** A certain physical quantity $q(x)$ can be represented by the series $\sum_{n=0}^{\infty} c_n x^n$ in $0 \leqslant x < x_0$, but the series diverges for $x > x_0$. Describe the Euler transformation to a new series which may enable $q(x)$ to be computed for $x > x_0$. Give the first four terms of the new series.

Describe briefly the disadvantages of the method.

**(ii)** The series $\sum_1^{\infty} c_r$ has partial sums $S_n = \sum_1^n c_r$. Describe Shanks' method to approximate $S_n$ by

$$S_n = A + BC^n \ , \qquad (*)$$

giving expressions for $A, B$ and $C$.

Denote by $B_N$ and $C_N$ the values of $B$ and $C$ respectively derived from these expressions using $S_{N-1}, S_N$ and $S_{N+1}$ for some fixed $N$. Now let $A^{(n)}$ be the value of $A$ obtained from $(*)$ with $B = B_N, C = C_N$. Show that, if $|C_N| < 1$,

$$\sum_1^{\infty} c_r = \lim_{n \to \infty} A^{(n)} \ .$$

If, in fact, the partial sums satisfy

$$S_n = a + \alpha c^n + \beta d^n \ ,$$

with $1 > |c| > |d|$, show that

$$A^{(n)} = A + \gamma d^n + o(d^n) \ ,$$

where $\gamma$ is to be found.

**18K Nonlinear Waves**

**(i)** Establish two conservation laws for the MKdV equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

State sufficient boundary conditions that $u$ should satisfy for the conservation laws to be valid.

**(ii)** The equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V\right) = 0$$

models traffic flow on a single-lane road, where $\rho(x,t)$ represents the density of cars, and $V$ is a given function of $\rho$. By considering the rate of change of the integral

$$\int_a^b \rho \, dx,$$

show that $V$ represents the velocity of the cars.

Suppose now that $V = 1 - \rho$ (in suitable units), and that $0 \leqslant \rho \leqslant 1$ everywhere. Assume that a queue is building up at a traffic light at $x = 1$, so that, when the light turns green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0 & \text{for } \; x < 0 \;\text{ and }\; x > 1 \\ x & \text{for } \; 0 \leqslant x < 1. \end{cases}$$

For this problem, find and sketch the characteristics in the $(x,t)$ plane, for $t > 0$, paying particular attention to those emerging from the point $(1,0)$. Show that a shock forms at $t = \frac{1}{2}$. Find the density of cars $\rho(x,t)$ for $0 < t < \frac{1}{2}$, and all $x$.

**19K Numerical Analysis**

**(i)** Define $m$-step BDF (backward differential formula) methods for the numerical solution of ordinary differential equations and derive explicitly their coefficients.

**(ii)** Prove that the linear stability domain of the two-step BDF method includes the interval $(-\infty, 0)$.