UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 8 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

Let f be a mapping of a metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ for all distinct x, y in X.

Show that $f(x)$ and $d(x, f(x))$ are continuous functions of x.

Now suppose that (X, d) is compact and let

$$h = \inf_{x \in X} d(x, f(x)).$$

Show that we cannot have $h > 0$.

[*You may assume that a continuous function on a compact metric space is bounded and attains its bounds.*]

Deduce that f possesses a fixed point, and that it is unique.

2H Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1 - x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0, \quad n = 0, 1, \ldots, \ -1 \leqslant x \leqslant 1.$$

Show that $R_n(x) = P_n'(x)$ obeys an equation which can be recast in Sturm–Liouville form and has the eigenvalue $(n-1)(n+2)$. What is the orthogonality relation for $R_n(x), R_m(x)$ for $n \neq m$?

3D Statistics

Consider the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

$i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are given constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, with σ^2 unknown.

Find the least squares estimator $\widehat{\beta}$ of β. State, without proof, the distribution of $\widehat{\beta}$ and describe how you would test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$, where β_0 is given.

4A Further Analysis

Let $f(z)$ be analytic in the disc $|z| < R$. Assume the formula

$$f'(z_0) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f(z)\,dz}{(z-z_0)^2}, \quad 0 \leqslant |z_0| < r < R.$$

By combining this formula with a complex conjugate version of Cauchy's Theorem, namely

$$0 = \int\limits_{|z|=r} \overline{f(z)}\,d\bar{z},$$

prove that

$$f'(0) = \frac{1}{\pi r} \int_0^{2\pi} u(\theta) e^{-i\theta} d\theta,$$

where $u(\theta)$ is the real part of $f(re^{i\theta})$.

5D Optimization

Explain what is meant by a two-person zero-sum game with payoff matrix $A = [a_{ij}]$. Write down a set of sufficient conditions for a pair of strategies to be optimal for such a game.

A fair coin is tossed and the result is shown to player I, who must then decide to 'pass' or 'bet'. If he passes, he must pay player II £1. If he bets, player II, who does not know the result of the coin toss, may either 'fold' or 'call the bet'. If player II folds, she pays player I £1. If she calls the bet and the toss was a head, she pays player I £2; if she calls the bet and the toss was a tail, player I must pay her £2.

Formulate this as a two-person zero-sum game and find optimal strategies for the two players. Show that the game has value $\frac{1}{3}$.

[*Hint: Player I has four possible moves and player II two.*]

6C Linear Mathematics

Find the Jordan normal form J of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and determine both the characteristic and the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J (the Jordan normal form of M) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

7G Fluid Dynamics

Starting from the Euler equation, derive the *vorticity equation* for the motion of an inviscid incompressible fluid under a conservative body force, and give a physical interpretation of each term in the equation. Deduce that in a flow field of the form $\mathbf{u} = (u(x,y,t), v(x,y,t), 0)$ the vorticity of a material particle is conserved.

Find the vorticity for such a flow in terms of the stream function ψ. Deduce that if the flow is steady, there must be a function f such that

$$\nabla^2\psi = f(\psi)\ .$$

8F Complex Methods

Consider a conformal mapping of the form

$$f(z) = \frac{a+bz}{c+dz}, \quad z \in \mathbb{C}\ ,$$

where $a, b, c, d \in \mathbb{C}$, and $ad \neq bc$. You may assume $b \neq 0$. Show that any such $f(z)$ which maps the unit circle onto itself is necessarily of the form

$$f(z) = e^{i\psi}\frac{a+z}{1+\bar{a}z}\cdot$$

[*Hint: Show that it is always possible to choose* $b = 1$.]

9F Special Relativity

What is Einstein's principle of relativity?

Show that a spherical shell expanding at the speed of light, $\mathbf{x}^2 = c^2t^2$, in one coordinate system $(t, \mathbf{x})$, is still spherical in a second coordinate system $(t', \mathbf{x}')$ defined by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{u}{c}x\right), \\ x' &= \gamma(x - ut), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$. Why do these equations define a Lorentz transformation?

SECTION II

10A Analysis II

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions on a closed interval $[a,b]$. Prove that, if for every $x \in [a,b]$, the sequence $\{f_n(x)\}$ is monotonic in n, and if all the functions $f_n,\ n=1,2,\ldots,$ and $f=\lim f_n$ are continuous, then $f_n \to f$ uniformly on $[a,b]$.

By considering a suitable sequence of functions $\{f_n\}$ on $[0,1)$, show that if the interval is not closed but all other conditions hold, the conclusion of the theorem may fail.

11H Methods

A curve $y(x)$ in the xy-plane connects the points $(\pm a, 0)$ and has a fixed length $l,\ 2a < l < \pi a$. Find an expression for the area A of the surface of the revolution obtained by rotating $y(x)$ about the x-axis.

Show that the area A has a stationary value for

$$y = \frac{1}{k}(\cosh kx - \cosh ka),$$

where k is a constant such that

$$lk = 2\sinh ka.$$

Show that the latter equation admits a unique positive solution for k.

12D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, where μ and σ^2 are unknown.

Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2, based on $X_1, \ldots, X_n$. Show that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent, and derive their distributions.

Suppose now it is intended to construct a "prediction interval" $I(X_1, \ldots, X_n)$ for a future, independent, $N(\mu, \sigma^2)$ random variable X_0. We require

$$P\Big\{X_0 \in I(X_1, \ldots, X_n)\Big\} = 1 - \alpha,$$

with the probability over the *joint* distribution of $X_0, X_1, \ldots, X_n$.

Let

$$I_\gamma(X_1, \ldots, X_n) = \left(\widehat{\mu} - \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}},\ \widehat{\mu} + \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}}\right).$$

By considering the distribution of $(X_0 - \widehat{\mu})/(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}})$, find the value of γ for which $P\{X_0 \in I_\gamma(X_1, \ldots, X_n)\} = 1 - \alpha$.

13B Further Analysis

Let $\Delta^* = \{z : 0 < |z| < r\}$ be a punctured disc, and f an analytic function on Δ^*. What does it mean to say that f has the origin as (i) a removable singularity, (ii) a pole, and (iii) an essential singularity? State criteria for (i), (ii), (iii) to occur, in terms of the Laurent series for f at 0.

Suppose now that the origin is an essential singularity for f. Given any $w \in \mathbb{C}$, show that there exists a sequence (z_n) of points in Δ^* such that $z_n \to 0$ and $f(z_n) \to w$. [*You may assume the fact that an isolated singularity is removable if the function is bounded in some open neighbourhood of the singularity.*]

State the Open Mapping Theorem. Prove that if f is analytic and injective on Δ^*, then the origin cannot be an essential singularity. By applying this to the function $g(1/z)$, or otherwise, deduce that if g is an injective analytic function on $\mathbb{C}$, then g is linear of the form $az + b$, for some non-zero complex number a. [*Here, you may assume that g injective implies that its derivative g' is nowhere vanishing.*]

14D Optimization

Dumbledore Publishers must decide how many copies of the best-selling "History of Hogwarts" to print in the next two months to meet demand. It is known that the demands will be for 40 thousand and 60 thousand copies in the first and second months respectively, and these demands must be met on time. At the beginning of the first month, a supply of 10 thousand copies is available, from existing stock. During each month, Dumbledore can produce up to 40 thousand copies, at a cost of 400 galleons per thousand copies. By having employees work overtime, up to 150 thousand additional copies can be printed each month, at a cost of 450 galleons per thousand copies. At the end of each month, after production and the current month's demand has been satisfied, a holding cost of 20 galleons per thousand copies is incurred.

Formulate a transportation problem, with 5 supply points and 3 demand points, to minimize the sum of production and holding costs during the two month period, and solve it.

[*You may assume that copies produced during a month can be used to meet demand in that month.*]

15C Linear Mathematics

Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that AB and BA have the same characteristic polynomial. [*Hint: Look at* $\det(CBC - xC)$ *for* $C = A + yI$*, where* x *and* y *are scalar variables.*]

Show by example that AB and BA need not have the same minimal polynomial.

Suppose that AB is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of BA must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.

16G Fluid Dynamics

A long straight canal has rectangular cross-section with a horizontal bottom and width $w(x)$ that varies slowly with distance x downstream. Far upstream, w has a constant value W, the water depth has a constant value H, and the velocity has a constant value U. Assuming that the water velocity is steady and uniform across the channel, use mass conservation and Bernoulli's theorem, which should be stated carefully, to show that the water depth $h(x)$ satisfies

$$\left(\frac{W}{w}\right)^2 = \left(1+\frac{2}{F}\right)\left(\frac{h}{H}\right)^2 - \frac{2}{F}\left(\frac{h}{H}\right)^3 \quad \text{where} \quad F = \frac{U^2}{gH} \ .$$

Deduce that for a given value of F, a flow of this kind can exist only if $w(x)$ is everywhere greater than or equal to a critical value w_c, which is to be determined in terms of F.

Suppose that $w(x) > w_c$ everywhere. At locations where the channel width exceeds W, determine graphically, or otherwise, under what circumstances the water depth exceeds H.

17F Complex Methods

State Jordan's Lemma.

Consider the integral

$$I = \oint_C dz \frac{z\ \sin(xz)}{(a^2+z^2)\sin \pi z},$$

for real x and a. The rectangular contour C runs from $+\infty + i\epsilon$ to $-\infty + i\epsilon$, to $-\infty - i\epsilon$, to $+\infty - i\epsilon$ and back to $+\infty + i\epsilon$, where ϵ is infinitesimal and positive. Perform the integral in two ways to show that

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin nx}{a^2+n^2} = -\pi \frac{\sinh ax}{\sinh a\pi},$$

for $|x| < \pi$.

18F Special Relativity

A particle of mass M is at rest at $x = 0$, in coordinates (t, x). At time $t = 0$ it decays into two particles A and B of equal mass $m < M/2$. Assume that particle A moves in the *negative* x direction.

(a) Using relativistic energy and momentum conservation compute the energy, momentum and velocity of both particles A and B.

(b) After a proper time τ, measured in its own rest frame, particle A decays. Show that the spacetime coordinates of this event are

$$t = \frac{M}{2m}\tau,$$
$$x = -\frac{MV}{2m}\tau,$$

where $V = c\sqrt{1 - 4(m/M)^2}$.

END OF PAPER