UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 8 June 2001 9 to 12

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, . . . , H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

Define what is meant by a norm on a real vector space.

(a) Prove that two norms on a vector space (not necessarily finite-dimensional) give rise to equivalent metrics if and only if they are Lipschitz equivalent.

(b) Prove that if the vector space V has an inner product, then for all $x, y \in V$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

in the induced norm.

Hence show that the norm on $\mathbb{R}^2$ defined by $\|x\| = \max(|x_1|, |x_2|)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, cannot be induced by an inner product.

2G Methods

Laplace's equation in the plane is given in terms of plane polar coordinates r and θ in the form

$$\nabla^2 \phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

In each of the cases

$$\text{(i)} \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad \text{(ii)} \quad 1 \leqslant r < \infty,$$

find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \leqslant r \leqslant b$ with the boundary conditions

$$\phi = 1 \quad \text{on} \quad r = a \quad \text{for} \quad \text{all} \quad \theta,$$

$$\phi = 2 \quad \text{on} \quad r = b \quad \text{for} \quad \text{all} \quad \theta.$$

3B Further Analysis

State a version of Rouché's Theorem. Find the number of solutions (counted with multiplicity) of the equation

$$z^4 = a(z-1)(z^2-1) + \tfrac{1}{2}$$

inside the open disc $\{z : |z| < \sqrt{2}\}$, for the cases $a = \frac{1}{3}, 12$ and 5.

[*Hint: For the case* $a = 5$*, you may find it helpful to consider the function* $(z^2-1)(z-2)(z-3)$.]

4B Geometry

State and prove the Gauss–Bonnet theorem for the area of a spherical triangle.

Suppose **D** is a regular dodecahedron, with centre the origin. Explain how each face of **D** gives rise to a spherical pentagon on the 2-sphere S^2. For each such spherical pentagon, calculate its angles and area.

5D Optimization

Let $a_1, \ldots, a_n$ be given constants, not all equal.

Use the Lagrangian sufficiency theorem, which you should state clearly, without proof, to minimize $\sum_{i=1}^n x_i^2$ subject to the two constraints $\sum_{i=1}^n x_i = 1, \sum_{i=1}^n a_i x_i = 0$.

6E Numerical Analysis

Given $f \in C^{n+1}[a,b]$, let the nth-degree polynomial p interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct. Given $x \in [a,b]$, find the error $f(x) - p(x)$ in terms of a derivative of f.

7C Linear Mathematics

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix M which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is W and which contains $(1,1,1,1,1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel, and image contained in W?

8G Fluid Dynamics

Inviscid incompressible fluid occupies the region $y > 0$, which is bounded by a rigid barrier along $y = 0$. At time $t = 0$, a line vortex of strength κ is placed at position (a,b). By considering the flow due to an image vortex at $(a,-b)$, or otherwise, determine the velocity potential in the fluid.

Derive the position of the original vortex at time $t > 0$.

9B Quadratic Mathematics

Let A be the Hermitian matrix

$$\begin{pmatrix} 1 & i & 2i \\ -i & 3 & -i \\ -2i & i & 5 \end{pmatrix}.$$

Explaining carefully the method you use, find a diagonal matrix D with **rational** entries, and an invertible (complex) matrix T such that $T^*DT = A$, where T^* here denotes the conjugated transpose of T.

Explain briefly why we cannot find T, D as above with T unitary.

[*You may assume that if a monic polynomial $t^3 + a_2t^2 + a_1t + a_0$ with integer coefficients has all its roots rational, then all its roots are in fact integers.*]

10F Special Relativity

A particle of rest mass m and four-momentum $P = (E/c, \mathbf{p})$ is detected by an observer with four-velocity U, satisfying $U \cdot U = c^2$, where the product of two four-vectors $P = (p^0, \mathbf{p})$ and $Q = (q^0, \mathbf{q})$ is $P \cdot Q = p^0q^0 - \mathbf{p} \cdot \mathbf{q}$.

Show that the speed of the detected particle in the observer's rest frame is

$$v = c\sqrt{1 - \frac{P \cdot Pc^2}{(P \cdot U)^2}}.$$

SECTION II

11A Analysis II

Prove that if all the partial derivatives of $f : \mathbb{R}^p \to \mathbb{R}$ (with $p \geqslant 2$) exist in an open set containing $(0, 0, \ldots, 0)$ and are continuous at this point, then f is differentiable at $(0, 0, \ldots, 0)$.

Let

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$f(x, y) = g(x) + g(y).$$

At which points of the plane is the partial derivative f_x continuous?

At which points is the function $f(x, y)$ differentiable? Justify your answers.

12H Methods

Find the Fourier sine series representation on the interval $0 \leqslant x \leqslant l$ of the function

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant b, \\ 0, & b < x \leqslant l. \end{cases}$$

The motion of a struck string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for} \quad 0 \leqslant x \leqslant l \quad \text{and} \quad t \geqslant 0,$$

subject to boundary conditions $y = 0$ at $x = 0$ and $x = l$ for $t \geqslant 0$, and to the initial conditions $y = 0$ and $\dfrac{\partial y}{\partial t} = \delta(x - \dfrac{1}{4}l)$ at $t = 0$.

Obtain the solution $y(x, t)$ for this motion. Evaluate $y(x, t)$ for $t = \frac{1}{2}l/c$, and sketch it clearly.

13B Further Analysis

If X and Y are topological spaces, describe the open sets in the *product topology* on $X \times Y$. If the topologies on X and Y are induced from metrics, prove that the same is true for the product.

What does it mean to say that a topological space is *compact*? If the topologies on X and Y are compact, prove that the same is true for the product.

14B Geometry

Describe the hyperbolic lines in the upper half-plane model H of the hyperbolic plane. The group $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts on H via Möbius transformations, which you may assume are isometries of H. Show that G acts transitively on the hyperbolic lines. Find explicit formulae for the reflection in the hyperbolic line L in the cases (i) L is a vertical line $x = a$, and (ii) L is the unit semi-circle with centre the origin. Verify that the composite T of a reflection of type (ii) followed afterwards by one of type (i) is given by $T(z) = -z^{-1} + 2a$.

Suppose now that L_1 and L_2 are distinct hyperbolic lines in the hyperbolic plane, with R_1, R_2 denoting the corresponding reflections. By considering different models of the hyperbolic plane, or otherwise, show that

(a) R_1R_2 has infinite order if L_1 and L_2 are parallel or ultraparallel, and

(b) R_1R_2 has finite order if and only if L_1 and L_2 meet at an angle which is a rational multiple of π.

15D Optimization

Consider the following linear programming problem,

$$\begin{array}{ll} \text{minimize} & (3-p)x_1 + px_2 \\ \text{subject to} & \begin{array}{rl} 2x_1 + x_2 & \geqslant 8 \\ x_1 + 3x_2 & \geqslant 9 \\ x_1 & \leqslant 6 \\ x_1,\ x_2 & \geqslant 0. \end{array} \end{array}$$

Formulate the problem in a suitable way for solution by the two-phase simplex method.

Using the two-phase simplex method, show that if $2 \leqslant p \leqslant \frac{9}{4}$ then the optimal solution has objective function value $9 - p$, while if $\frac{9}{4} < p \leqslant 3$ the optimal objective function value is $18 - 5p$.

16E Numerical Analysis

Let the monic polynomials p_n, $n \geqslant 0$, be orthogonal with respect to the weight function $w(x) > 0$, $a < x < b$, where the degree of each p_n is exactly n.

(a) Prove that each p_n, $n \geqslant 1$, has n distinct zeros in the interval (a, b).

(b) Suppose that the p_n satisfy the three-term recurrence relation

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n^2 p_{n-2}(x), \quad n \geqslant 2,$$

where $p_0(x) \equiv 1$, $p_1(x) = x - a_1$. Set

$$A_n = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}, \quad n \geqslant 2.$$

Prove that $p_n(x) = \det(xI - A_n)$, $n \geqslant 2$, and deduce that all the eigenvalues of A_n reside in (a, b).

17C Linear Mathematics

Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of V, i.e. $\alpha^m = 0$ for some positive integer m. Prove that α can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). [*You may wish to consider the subspaces* $\ker(\alpha^j)$ *for* $j = 1, \ldots, m$.]

Show that if α is nilpotent, then $\alpha^n = 0$ where n is the dimension of V. Give an example of a 4×4 matrix M such that $M^4 = 0$ but $M^3 \neq 0$.

Let A be a nilpotent matrix and I the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I+A)(I+B)$ if A and B are nilpotent? Justify your answer.

18G Fluid Dynamics

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

A circular cylinder of radius a is immersed in unbounded inviscid fluid of uniform density ρ. The cylinder moves in a prescribed direction perpendicular to its axis, with speed U. Use cylindrical polar coordinates, with the direction $\theta = 0$ parallel to the direction of the motion, to find the velocity potential in the fluid.

If U depends on time t and gravity is negligible, determine the pressure field in the fluid at time t. Deduce the fluid force per unit length on the cylinder.

[In cylindrical polar coordinates, $\nabla\phi = \dfrac{\partial\phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$.]

19B Quadratic Mathematics

Let J_1 denote the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that T is a 2×2 upper-triangular real matrix with strictly positive diagonal entries and that ${J_1}^{-1}TJ_1T^{-1}$ is orthogonal. Verify that $J_1T = TJ_1$.

Prove that any real invertible matrix A has a decomposition $A = BC$, where B is an orthogonal matrix and C is an upper-triangular matrix with strictly positive diagonal entries.

Let A now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let K denote the $2n \times 2n$ matrix with n copies of J_1 on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1}CKC^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1}CKC^{-1}$ are zero, apart from n orthogonal 2×2 blocks $E_1, \ldots, E_n$ along the diagonal. Show that each E_i has the form ${J_1}^{-1}C_iJ_1{C_i}^{-1}$, for some 2×2 upper-triangular matrix C_i with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[*Hint: The invertible $2n \times 2n$ matrices S with 2×2 blocks $S_1, \ldots, S_n$ along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.*]

20F Quantum Mechanics

A quantum system has a complete set of orthonormalised energy eigenfunctions $\psi_n(x)$ with corresponding energy eigenvalues $E_n,\ n = 1, 2, 3, \ldots$.

(a) If the time-dependent wavefunction $\psi(x, t)$ is, at $t = 0$,

$$\psi(x, 0) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

determine $\psi(x, t)$ for all $t > 0$.

(b) A linear operator $\mathcal{S}$ acts on the energy eigenfunctions as follows:

$$\begin{aligned} \mathcal{S}\psi_1 &= 7\psi_1 + 24\psi_2, \\ \mathcal{S}\psi_2 &= 24\psi_1 - 7\psi_2, \\ \mathcal{S}\psi_n &= 0, \quad n \geqslant 3. \end{aligned}$$

Find the eigenvalues of $\mathcal{S}$. At time $t = 0,\ \mathcal{S}$ is measured and its lowest eigenvalue is found. At time $t > 0,\ \mathcal{S}$ is measured again. Show that the probability for obtaining the lowest eigenvalue again is

$$\frac{1}{625}\left(337 + 288\cos(\omega t)\right),$$

where $\omega = (E_1 - E_2)/\hbar$.

END OF PAPER