UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 6 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

Define uniform continuity for functions defined on a (bounded or unbounded) interval in $\mathbb{R}$.

Is it true that a real function defined and uniformly continuous on $[0,1]$ is necessarily bounded?

Is it true that a real function defined and with a bounded derivative on $[1,\infty)$ is necessarily uniformly continuous there?

Which of the following functions are uniformly continuous on $[1,\infty)$:

(i) x^2;

(ii) $\sin(x^2)$;

(iii) $\dfrac{\sin x}{x}$?

Justify your answers.

2H Methods

The even function $f(x)$ has the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $-\pi \leqslant x \leqslant \pi$. Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2.$$

Find the Fourier cosine series of x^2 in the same interval, and show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a two-dimensional sufficient statistic for μ, quoting carefully, without proof, any result you use.

What is the maximum likelihood estimator of μ?

4B Geometry

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. What are the geodesics passing through the origin? Show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$.

Write down an isometry between the upper half-plane model H of the hyperbolic plane and the disc model Δ, under which $i \in H$ corresponds to $0 \in \Delta$. Show that the hyperbolic circle of radius ρ centred on i in H is a Euclidean circle with centre $i \cosh \rho$ and of radius $\sinh \rho$.

5C Linear Mathematics

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix} x & 1 & 1 \\ 1-x & 0 & -1 \\ 2 & 2x & 1 \end{pmatrix}$$

is invertible. Determine the rank of M as a function of x. Find the adjugate and hence the inverse of M for general x.

6G Fluid Dynamics

Determine the pressure at a depth z below the surface of a static fluid of density ρ subject to gravity g. A rigid body having volume V is fully submerged in such a fluid. Calculate the buoyancy force on the body.

An iceberg of uniform density ρ_I is observed to float with volume V_I protruding above a large static expanse of seawater of density ρ_w. What is the total volume of the iceberg?

7E Complex Methods

State the Cauchy integral formula.

Assuming that the function $f(z)$ is analytic in the disc $|z| < 1$, prove that, for every $0 < r < 1$, it is true that

$$\frac{d^n f(0)}{dz^n} = \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, \qquad n = 0, 1, \ldots.$$

[*Taylor's theorem may be used if clearly stated.*]

8B Quadratic Mathematics

Let $q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Define what is meant by the *discriminant* d of q, and show that q is positive-definite if and only if $a > 0 > d$. Define what it means for the form q to be *reduced*. For any integer $d < 0$, we define the class number $h(d)$ to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant d. Show that $h(d)$ is always finite (for negative d). Find $h(-39)$, and exhibit the corresponding reduced forms.

9F Quantum Mechanics

A quantum mechanical particle of mass m and energy E encounters a potential step,

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geqslant 0. \end{cases}$$

Calculate the probability P that the particle is reflected in the case $E > V_0$.

If V_0 is positive, what is the limiting value of P when E tends to V_0? If V_0 is negative, what is the limiting value of P as V_0 tends to $-\infty$ for fixed E?

SECTION II

10A Analysis II

Show that each of the functions below is a metric on the set of functions $x(t) \in C[a,b]$:

$$d_1(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

$$d_2(x,y) = \left\{ \int_a^b |x(t) - y(t)|^2 dt \right\}^{1/2}.$$

Is the space complete in the d_1 metric? Justify your answer.

Show that the set of functions

$$x_n(t) = \begin{cases} 0, & -1 \leqslant t < 0 \\ nt, & 0 \leqslant t < 1/n \\ 1, & 1/n \leqslant t \leqslant 1 \end{cases}$$

is a Cauchy sequence with respect to the d_2 metric on $C[-1,1]$, yet does not tend to a limit in the d_2 metric in this space. Hence, deduce that this space is not complete in the d_2 metric.

11H Methods

Use the substitution $y = x^p$ to find the general solution of

$$\mathcal{L}_x y \equiv \frac{d^2y}{dx^2} - \frac{2}{x^2} y = 0.$$

Find the Green's function $G(x,\xi)$, $0 < \xi < \infty$, which satisfies

$$\mathcal{L}_x G(x,\xi) = \delta(x-\xi)$$

for $x > 0$, subject to the boundary conditions $G(x,\xi) \to 0$ as $x \to 0$ and as $x \to \infty$, for each fixed ξ.

Hence, find the solution of the equation

$$\mathcal{L}_x y = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & x > 1, \end{cases}$$

subject to the same boundary conditions.

Verify that both forms of your solution satisfy the appropriate equation and boundary conditions, and match at $x = 1$.

12D Statistics

What is a *simple hypothesis*? Define the terms *size* and *power* for a test of one simple hypothesis against another.

State, without proof, the Neyman–Pearson lemma.

Let X be a **single** random variable, with distribution F. Consider testing the null hypothesis $H_0 : F$ is standard normal, $N(0,1)$, against the alternative hypothesis $H_1 : F$ is double exponential, with density $\frac{1}{4}e^{-|x|/2}, x \in \mathbb{R}$.

Find the test of size $\alpha, \alpha < \frac{1}{4}$, which maximises power, and show that the power is $e^{-t/2}$, where $\Phi(t) = 1 - \alpha/2$ and Φ is the distribution function of $N(0,1)$.

[*Hint: if* $X \sim N(0,1),\ P(|X| > 1) = 0.3174$.]

13B Geometry

Describe geometrically the stereographic projection map ϕ from the unit sphere S^2 to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and find a formula for ϕ. Show that any rotation of S^2 about the z-axis corresponds to a Möbius transformation of $\mathbb{C}_\infty$. You are given that the rotation of S^2 defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

corresponds under ϕ to a Möbius transformation of $\mathbb{C}_\infty$; deduce that any rotation of S^2 about the x-axis also corresponds to a Möbius transformation.

Suppose now that $u, v \in \mathbb{C}$ correspond under ϕ to distinct points $P, Q \in S^2$, and let d denote the angular distance from P to Q on S^2. Show that $-\tan^2(d/2)$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in some order (which you should specify). [*You may assume that the cross-ratio is invariant under Möbius transformations.*]

14C Linear Mathematics

(a) Find a matrix M over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x-2)^3(x+1)^2$. Furthermore find two matrices M_1 and M_2 over $\mathbb{C}$ which have the same characteristic polynomial, $(x-3)^5(x-1)^2$, and the same minimal polynomial, $(x-3)^2(x-1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, M_3, neither conjugate to M_1 nor to M_2? Justify your answer.

(b) Suppose A is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x-\lambda_1)(x-\lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which A defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota)$, where ι is the identity.

[*Hint: Express* $v \in V$ *in terms of* $(\alpha - \lambda_1\iota)(v)$ *and* $(\alpha - \lambda_2\iota)(v)$.]

Now suppose that A has minimal polynomial $(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)$ for distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \ldots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

15G Fluid Dynamics

A fluid motion has velocity potential $\phi(x, y, t)$ given by

$$\phi = \epsilon y \cos(x - t)$$

where ϵ is a constant. Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\nabla \cdot \mathbf{u}$.

The *time-average* of a quantity $\psi(x, y, t)$ is defined as $\frac{1}{2\pi}\int_0^{2\pi} \psi(x, y, t)\, dt$.

Show that the time-average of this velocity field at every point (x, y) is zero.

Write down an expression for the fluid acceleration and find the time-average acceleration at (x, y).

Suppose now that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at time $t = 0$ is marked with dye. Write down equations for its subsequent motion and verify that its position (x, y) at time $t > 0$ is given (correct to terms of order ϵ^2) as

$$x = \epsilon^2\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right),$$
$$y = \epsilon \sin t\ .$$

Deduce the time-average velocity of the dyed particle correct to this order.

16E Complex Methods

Let the function F be integrable for all real arguments x, such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \ ,$$

and assume that the series

$$f(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly for all $0 \leqslant \tau \leqslant 2\pi$.

Prove the Poisson summation formula

$$f(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{in\tau},$$

where $\hat{F}$ is the Fourier transform of F. [*Hint: You may show that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-imx} f(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imx} F(x) dx$$

or, alternatively, prove that f is periodic and express its Fourier expansion coefficients explicitly in terms of $\hat{F}$.]

Letting $F(x) = e^{-|x|}$, use the Poisson summation formula to evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \ .$$

17B Quadratic Mathematics

Let ϕ be a symmetric bilinear form on a finite dimensional vector space V over a field k of characteristic $\neq 2$. Prove that the form ϕ may be diagonalized, and interpret the rank r of ϕ in terms of the resulting diagonal form.

For ϕ a symmetric bilinear form on a real vector space V of finite dimension n, define the *signature* σ of ϕ, proving that it is well-defined. A subspace U of V is called *null* if $\phi|_U \equiv 0$; show that V has a null subspace of dimension $n - \frac{1}{2}(r + |\sigma|)$, but no null subspace of higher dimension.

Consider now the quadratic form q on $\mathbb{R}^5$ given by

$$2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).$$

Write down the matrix A for the corresponding symmetric bilinear form, and calculate $\det A$. Hence, or otherwise, find the rank and signature of q.

18F Quantum Mechanics

Consider a quantum-mechanical particle of mass m moving in a potential well,

$$V(x) = \begin{cases} 0, & -a < x < a, \\ \infty, & \text{elsewhere.} \end{cases}$$

(a) Verify that the set of normalised energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{1}{a}} \, \sin\left(\frac{n\pi(x+a)}{2a}\right), \qquad n = 1, 2, \ldots,$$

and evaluate the corresponding energy eigenvalues E_n.

(b) At time $t = 0$ the wavefunction for the particle is only nonzero in the positive half of the well,

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \, \sin\left(\frac{\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{elsewhere.} \end{cases}$$

Evaluate the expectation value of the energy, first using

$$\langle E \rangle = \int_{-a}^{a} \psi H \psi dx,$$

and secondly using

$$\langle E \rangle = \sum_n |a_n|^2 E_n,$$

where the a_n are the expansion coefficients in

$$\psi(x) = \sum_n a_n \psi_n(x).$$

Hence, show that

$$1 = \frac{1}{2} + \frac{8}{\pi^2} \sum_{p=0}^{\infty} \frac{(2p+1)^2}{[(2p+1)^2 - 4]^2} \; .$$

END OF PAPER