UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 5 June 2001 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I and at most* ***five*** *questions from Section II. In Section II no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A, B, C, D, E, F*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1F Algebra and Geometry

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 = 0$ if and only if $a = -d$ and $bc = -a^2$. Prove that $A^3 = 0$ if and only if $A^2 = 0$.

[*Hint: it is easy to check that* $A^2 - (a+d)A + (ad - bc)I = 0$.]

2D Algebra and Geometry

Show that the set of Möbius transformations of the extended complex plane $\mathbb{C} \cup \{\infty\}$ form a group. Show further that an arbitrary Möbius transformation can be expressed as the composition of maps of the form

$$f(z) = z + a, \quad g(z) = kz \text{ and } h(z) = 1/z\ .$$

3C Vector Calculus

For a real function $f(x, y)$ with $x = x(t)$ and $y = y(t)$ state the chain rule for the derivative $\frac{d}{dt} f(x(t), y(t))$.

By changing variables to u and v, where $u = \alpha(x)y$ and $v = y/x$ with a suitable function $\alpha(x)$ to be determined, find the general solution of the equation

$$x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y} \;=\; 6f\ .$$

4A Vector Calculus

Suppose that

$$u = y^2 \sin(xz) + xy^2 z \cos(xz), \quad v = 2xy \sin(xz), \quad w = x^2 y^2 \cos(xz).$$

Show that $u\,dx + v\,dy + w\,dz$ is an exact differential.

Show that

$$\int_{(0,0,0)}^{(\pi/2,1,1)} u\,dx + v\,dy + w\,dz \;=\; \frac{\pi}{2}.$$

SECTION II

5F Algebra and Geometry

Let A, B, C be 2×2 matrices, real or complex. Define the trace $\operatorname{tr} C$ to be the sum of diagonal entries $C_{11} + C_{22}$. Define the commutator $[A, B]$ to be the difference $AB - BA$. Give the definition of the eigenvalues of a 2×2 matrix and prove that it can have at most two distinct eigenvalues. Prove that

a) $\operatorname{tr}[A, B] = 0$,

b) $\operatorname{tr} C$ equals the sum of the eigenvalues of C,

c) if all eigenvalues of C are equal to 0 then $C^2 = 0$,

d) either $[A, B]$ is a diagonalisable matrix or the square $[A, B]^2 = 0$,

e) $[A, B]^2 = \alpha I$ where $\alpha \in \mathbb{C}$ and I is the unit matrix.

6E Algebra and Geometry

Define the notion of an *action* of a group G on a set X. Define *orbit* and *stabilizer*, and then, assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Show that the group of rotations of a cube has order 24.

7E Algebra and Geometry

State Lagrange's theorem. Use it to describe all groups of order p, where p is a fixed prime number.

Find all the subgroups of a fixed cyclic group $\langle x \rangle$ of order n.

8D Algebra and Geometry

(i) Let A_4 denote the alternating group of even permutations of four symbols. Let X be the 3-cycle (123) and P, Q be the pairs of transpositions $(12)(34)$ and $(13)(24)$. Find $X^3, P^2, Q^2, X^{-1}PX, X^{-1}QX$, and show that A_4 is generated by X, P and Q.

(ii) Let G and H be groups and let

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Show how to make $G \times H$ into a group in such a way that $G \times H$ contains subgroups isomorphic to G and H.

If D_n is the dihedral group of order n and C_2 is the cyclic group of order 2, show that D_{12} is isomorphic to $D_6 \times C_2$. Is the group D_{12} isomorphic to A_4?

9C Vector Calculus

Explain, with justification, how the nature of a critical (stationary) point of a function $f(\mathbf{x})$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$ if H is non-singular. What happens if H is singular?

Let $f(x,y) = (y-x^2)(y-2x^2) + \alpha x^2$. Find the critical points of f and determine their nature in the different cases that arise according to the values of the parameter $\alpha \in \mathbb{R}$.

10A Vector Calculus

State the rule for changing variables in a double integral.

Let D be the region defined by

$$\begin{cases} 1/x \leq y \leq 4x & \text{when } \frac{1}{2} \leq x \leq 1, \\ x \leq y \leq 4/x & \text{when } 1 \leq x \leq 2. \end{cases}$$

Using the transformation $u = y/x$ and $v = xy$, show that

$$\int_D \frac{4xy^3}{x^2+y^2}\, dxdy \;=\; \frac{15}{2} \ln \frac{17}{2}.$$

11B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \nabla\,\Omega\, dv \;=\; \int_S \Omega\, d\mathbf{S}\ . \qquad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x,y,z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \leq r \leq \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$ where a is a constant.

12B Vector Calculus

In $\mathbb{R}^3$ show that, within a closed surface S, there is at most one solution of Poisson's equation, $\nabla^2\phi = \rho$, satisfying the boundary condition on S

$$\alpha \frac{\partial \phi}{\partial n} + \phi = \gamma \ ,$$

where α and γ are functions of position on S, and α is everywhere non-negative.

Show that

$$\phi(x, y) = e^{\pm lx} \sin ly$$

are solutions of Laplace's equation $\nabla^2\phi = 0$ on $\mathbb{R}^2$.

Find a solution $\phi(x, y)$ of Laplace's equation in the region $0 < x < \pi$, $0 < y < \pi$ that satisfies the boundary conditions

$$\begin{array}{cccc}
\phi = 0 & \text{on} & 0 < x < \pi & y = 0 \\
\phi = 0 & \text{on} & 0 < x < \pi & y = \pi \\
\phi + \partial\phi/\partial n = 0 & \text{on} & x = 0 & 0 < y < \pi \\
\phi = \sin(ky) & \text{on} & x = \pi & 0 < y < \pi
\end{array}$$

where k is a positive integer. Is your solution the only possible solution?

END OF PAPER