UNIVERSITY OF CAMBRIDGE

Faculty of Mathematics

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COURSES IN PART IA

OF THE

MATHEMATICAL TRIPOS

This document contains a list of all the courses which are lectured in the first year of the Mathematical Tripos together with an informal description of each course. A formal syllabus is given in the Faculty booklet *Schedules for the Mathematical Tripos*

All Faculty documentation is available on the WWW (http://www.maths.cam.ac.uk/).

August 2, 2011

1 Structure of the Tripos

The Tripos consists of three parts (Part IA, Part IB and Part II) together with the optional Part III. Unlike programmes provided by most other UK universities, the Tripos is not modular: it is tightly structured, allowing no choice in the first year, but a wide choice in the third year; and examinations (especially in the second and third years) are cross-sectional, meaning that instead of each lecture course having a dedicated examination paper, each examination paper has questions on many lecture courses. The flexibility that this allows is considered by the Faculty to be one of the great strengths of the Tripos.

The decisions you need to make (after discussions with your Director of Studies) in the Mathematical Tripos are as follows.

1. In the first year you have to decide which of the two options to take: Pure and Applied Mathematics (Option (*a*)) or Mathematics with Physics (Option (*b*)). Within each of these options, you are expected to study all the lecture courses.

2. In the Michaelmas term of the first year, you have to decide whether to attend the non-examinable Mechanics course. This short course is intended for students who have taken fewer than three A-level modules in Mechanics (or the equivalent for students who took IB, Scottish Highers or other examinations).

3. In the Easter term of the first year, you have to decide which (if any) Part IB courses you wish to take. Most of them can also be taken in your second year.

4. In the second year and, especially, the third year, there is a wide range of courses, including CATAM, from which to choose.

Part IA options

There are two options: (a) Pure and Applied Mathematics; and (b) Mathematics with Physics. Technically, you do not have to decide which option to be examined in until the final examination entries are due in the Lent term, but clearly it would be sensible to make an early decision and stick to it.
Students offering Option (*b*) take Papers 1–3 of Part IA of the Mathematical Tripos and the Physics paper from the Natural Sciences Tripos instead of Paper 4. Students taking the different options are classed all together, the Mathematics with Physics option being indicated on the class list by a letter p.
At the end of the year, students taking option (b) have to decide whether to change tripos (to Natural Sciences) or continue with the Mathematical Tripos.

Option (b) is designed for students who have a strong interest in mathematics but who are likely to change to the Part IB of the Natural Sciences Tripos (Physics A / Physics B / Mathematics options) after the first year. It provides an excellent background for students who wish to study theoretical physics: their greater mathematical knowledge, compared with students who come to Physics through Part IA of the Natural Sciences Tripos, will be of enormous benefit. Option (*b*) includes all the Option (*a*) courses except for Numbers and Sets (Michaelmas Term) and Dynamics and Relativity (Lent term). These are replaced by the Physics course from the first year of the Natural Science Tripos, which starts at the beginning of the Michaelmas term and continues into the Easter term. There are also practical classes. The lecture timetable is arranged so that there is no clash between any Part IA lectures so if you decide on Mathematics with Physics you could still attend the lectures on Dynamics and Relativity, and Numbers and Sets.

Changes since last year

There have been no changes to the schedules for 2011/12.

Additional first-year work

Most students will find that there is enough mathematics in Part IA to keep them busy (or very busy!). For those who want something extra or something a bit different, the first choice should be the many excellent lectures provided by the student mathematics societies. Some students reach the end of their time at Cambridge and realise too late that they have missed the opportunity of hearing leading experts talking about some of the most important new ideas in mathematics and theoretical physics.

You are allowed to attend any lecture given in the University (provided it does not require limited laboratory or other facilities). It is worth looking through the Lecture List edition of the Reporter (see www.admin.cam.ac.uk/reporter/current/special) to see what is on offer: you may well something interesting and accessible in some other Tripos.

You could also attend lectures in Part IB or even Part II of the Mathematical Tripos in your first year. However, it is unlikely that you would have time to do supervision work. Two Part IB courses are timetabled so that they can be attended by first year students in the Lent term, namely Complex Methods and Groups, Rings and Modules. If you felt very comfortable with the workload in the Michaelmas term, you might consider attending one of these courses; however, your Director of Studies will probably advise you to concentrate on learning the Part IA courses thoroughly; that way, you would need less revision time in the Easter term and this would allow you to take the courses provided then (Metric and Topological Spaces, Variational Principles and Optimisation, or the non-examinable Concepts in Theoretical Physics).

Easter term courses

Four Part IB courses are lectured in the Easter terms: Optimisation, Metric and Topological Spaces, Variational Principles and the Computational Projects course. They are all examined in Part IB only. Lectures for Computational Projects have to be attended in the Easter term of the first year, but the other three courses can be taken either in the first year or in the second year.

Some of the material in Metric and Topological Spaces is important for the Part IB Complex Analysis course so if you are planning to take Complex Analysis rather than Complex Methods, it would therefore be a very good idea to take Metric and Topological Spaces in the Easter term of the first year. Similarly, Variational Principles provides important background material for many of the other applied courses in Part IB, so it is sensible to take this course in the Easter term of the first year.

Optimisation can be attended in the Easter term of your first or second year (or both). The advantage of taking it in your first year is that courses prepared well in advance of the examination sink in much better than courses prepared just before the examination.

For the Computational Projects course, you will be issued with a booklet describing the projects (probably in July). You should bear in mind that the computational work can be time-consuming; an early start would be sensible. The two core projects require no mathematical knowledge from Part IB courses so a good start can be made over the summer vacation. The faculty CATAM laboratory is usually open over the vacations, but check, just in case. Many colleges have suitable computing facilities which you can use at any time.

In addition to the courses listed above, there is a non-examinable course, Concepts in Theoretical Physics, in the Easter term, successfully pioneered in 2007/08. The intention is to give a flavour of some of the most important areas of modern theoretical physics. These 8 lectures should be of interest to all students.

2 Course descriptions: Part IA courses

Below, there is an informal description of each of the lecture courses given by the Faculty of Mathematics available in your first year. It ends with a summary of the learning outcomes of the course. The full learning outcome is that you should have understood the material described in the formal syllabuses given in the *Schedules of Lecture Courses for the Mathematical Tripos* and be able to apply it to the sort of problems that can be found on previous Tripos papers.

Vectors and Matrices — 24 lectures, Michaelmas term

The course starts with revision of complex numbers. It then introduces some more advanced ideas, including de Moivre's theorem which may be new to you. It moves on to generalise to higher (possible complex) dimensions the familiar idea of a vector. A very important tool, suffix notation, is used for vector algebra. This is followed by the application of vector methods to geometry.
The remainder of the course is taken up with matrices: algebraic manipulation; applications to solution of simultaneous equations; geometrical applications; and eigenvectors and eigenvalues. The material in this course is absolutely fundamental to nearly all areas of mathematics.

Learning outcomes

By the end of this course, you should:

- be able to manipulate complex numbers and be able to solve geometrical problems using complex numbers;
- be able to manipulate vectors in $\mathbb{R}^3$ (using suffix notation and summation convention where appropriate), and to solve geometrical problems using vectors;
- be able to manipulate matrices and determinants, and understand their relation to linear maps and systems of linear equations;
- be able to calculate eigenvectors and eigenvalues and understand their relation with diagonalisation of matrices, and canonical form.

Groups — 24 lectures, Michaelmas term

In university mathematics, *algebra* is the study of abstract systems of objects whose behaviour is governed by fixed rules (*axioms*). An example is the set of real numbers, which is governed by the rules of multiplication and addition. One of the simplest forms of abstract algebra is a group, which is roughly a set of objects and a rule for multiplying them together. Groups arise all over mathematics, particularly where there is symmetry.
The course introduces groups and their properties. The emphasis is on both the general theory and the many examples, such as groups of symmetries and groups of linear transformations.

Learning outcomes

By the end of this course, you should:

- be familiar with elementary properties of abstract groups, including the theory of mappings between groups;
- understand the group-theoretic perspective on symmetries in geometry.

Analysis I **24 lectures, Lent term**

Analysis is the rigorous investigation of calculus. You need to study analysis to have a firm foundation for techniques you already know, such as basic differentiation and integration. This not only allows you to understand exactly when these techniques can be used, but also allows you to generalise them to more complicated situations.
The sort of questions that you will be asking in this course are: 'what does it mean to say that a sequence or a function tends to a limit?'; 'what is the exact definition of a derivative or an integral?'; 'which functions can be differentiated and which can be integrated?'; 'how can these techniques be applied to functions on the complex plane?'; 'what conditions are needed to make Taylor's' series valid?'.
In later courses on analysis, differentiation and integration of functions of more than one variable are investigated.
In Analysis I, you will encounter the 'ϵ-δ' method (sometimes called 'epsilonics') of characterising the properties of functions. This is the basis of rigorous thought in mathematics, and will repay you handsomely for all the work you put into understanding it.

Learning outcomes

By the end of this course, you should:

- be able to apply the basic techniques of rigorous analysis and be familiar with examples of 'good behaviour' and 'bad-behaviour' in basic analysis;
- know the definition of a limit and be able to establish the convergence or divergence of simple real and complex sequences and series;
- understand the completeness of the real line and be able to derive the basic properties of continuous real-valued functions;
- be able to establish the rules for differentiation, and to prove and apply the mean value theorem;
- be acquainted with complex power series and be able to determine the radius of convergence in simple cases;
- know the definition of the Riemann integral, be able to test simple functions for integrability, and establish the rules for integration.

Vector Calculus **24 lectures, Lent term**

This course is about functions of more than one variable. It is an 'applied' course, meaning that you are expected to be able to apply techniques, but not necessarily to prove rigorously that they work – that will come in future analysis courses.
In the first part of the course, the idea of integration is extended from $\mathbb{R}$ to $\mathbb{R}^2$ and $\mathbb{R}^3$ (with an obvious extension to higher dimensions): integrals along the x-axis are replaced by integrals over curves, surfaces and volumes.
Then the idea of differentiation is extended to vectors (div, grad and curl), which is a basic tool in many areas of theoretical physics (such as electromagnetism and fluid dynamics).
Two important theorems are introduced, namely the divergence theorem and Stokes's theorem; in both cases, an integral over a region (in $\mathbb{R}^3$ and in $\mathbb{R}^2$, respectively) is converted to an integral over the boundary of the region.
All the previous ideas are then applied to Laplace's equation $\nabla^2\phi = 0$, which is one of the most important equations in all of mathematics and physics.
Finally, the notion of a vector is generalised to that of a *tensor*. A vector can be thought of as a 3×1 matrix that carries physical information: namely, magnitude and direction. This information is preserved when the axes are rotated only if the components change according to a certain rule. Very often, it is necessary to describe physical quantities using a 3×3 matrix (or even a $3 \times 3 \times 3 \cdots$ 'matrix'). Such a quantity is called a tensor if its components transform according to a certain rule when the axes are rotated. This rule means that the physical information in the tensor (essentially the eigenvalues) is preserved.

Learning outcomes

By the end of this course, you should:

- be able to manipulate, and solve problems using, vector operators;
- be able to calculate line, surface and volume integrals in $\mathbb{R}^3$, using Stokes theorem and the divergence theorem where appropriate;
- be able to solve Laplace's equation in simple cases, and be able to prove standard uniqueness theorems for Laplace's and related equations.
- understand the notion of a tensor and the general properties of tensors in simple cases.

Differential Equations — 24 lectures, Michaelmas term

The main aim is to develop the skill of representing real (physical or biological) situations by means of difference or differential equations. The course follows smoothly from the A-level syllabus starting with revision of differentiation and integration.

A particularly important sort of differential equation is one which is linear and has constant coefficients. These equations are unusual in that they can be solved exactly (the solutions are exponential or trigonometric functions). Many of the equations of physics are of this sort: the equations governing radioactive decay, Maxwell's equations for electromagnetism and the Schrödinger equation in quantum mechanics, for example.

In other cases, it is useful to try to represent solutions which cannot be obtained explicitly by means of phase-plane diagrams. Sometimes a particular solution describing some important situation is known although the general solution is not. In this case, it is often important to determine whether this solution is typical, or whether a small change in the conditions will lead to a very different solution. In the latter case, the solution is said to be unstable. This property is determined by linearising the original equation to obtain an equation with constant coefficients of the sort discussed above. Sometimes, the solutions are so unstable that they are called *chaotic*.

The very important idea of partial differentiation is introduced in the last half of the course. This is the analogue of familiar differentiation to functions which depend on more than one variable. The approach is mainly geometrical and one of the applications is determining the stationary points of, for example, a function that gives height above sea-level and classifying them into maxima (mountain peaks), minima (valley bottoms) and saddle points (cols or passes).

Learning outcomes

By the end of this course, you should:

- understand the theory of, and be able to solve (in simple cases), linear difference and differential equations, and standard types of non-linear equations;
- calculate partial derivatives and use the chain rule;
- find and classify stationary points of functions of more than one variable;
- be able to investigate the stability of solutions of difference and differential equations.

Probability — 24 lectures, Lent term

From its origin in games of chance and the analysis of experimental data, probability theory has developed into an area of mathematics with many varied applications in physics, biology and business.

The course introduces the basic ideas of probability and should be accessible to students who have no previous experience of probability or statistics. While developing the underlying theory, the course should strengthen students' general mathematical background and manipulative skills by its use of the axiomatic approach. There are links with other courses, in particular Vectors and Matrices, the elementary combinatorics of Numbers and Sets, the difference equations of Differential Equations and calculus of Vector Calculus and Analysis. Students should be left with a sense of the power of mathematics in relation to a variety of application areas.

After a discussion of basic concepts (including conditional probability, Bayes' formula, the binomial and Poisson distributions, and expectation), the course studies random walks, branching processes, geometric probability, simulation, sampling and the central limit theorem. Random walks can be used, for example, to represent the movement of a molecule of gas or the fluctuations of a share price; branching processes have applications in the modelling of chain reactions and epidemics. Through its treatment of discrete and continuous random variables, the course lays the foundation for the later study of statistical inference.

Learning outcomes

By the end of this course, you should:

- understand the basic concepts of probability theory, including independence, conditional probability, Bayes' formula, expectation, variance and generating functions;
- be familiar with the properties of commonly-used distribution functions for discrete and continuous random variables;
- understand and be able to apply the central limit theorem.
- be able to apply the above theory to 'real world' problems, including random walks and branching processes.

Numbers and Sets — 24 lectures, Michaelmas term

This course is concerned not so much with teaching you new parts of mathematics as with explaining how the language of mathematical arguments is used. We will use simple Mathematics to develop an understanding of how results are established.

Because you will exploring a broader and more intricate range of mathematical ideas at University, you will need to develop greater skills in understanding arguments and in formulating your own. These arguments are usually constructed in a careful, logical way as proofs of propositions. We begin with clearly stated and plausible assumptions or *axioms* and then develop a more and more complex theory from them. The course, and the lecturer, will have succeeded if you finish the course able to construct valid arguments of your own and to criticise those that are presented to you. Example sheets and supervisions will play a key role in achieving this. These skills will form the basis for the later courses, particularly those devoted to Pure Mathematics.

In order to give examples of arguments, we will take two topics: sets and numbers. Set theory provides a basic vocabulary for much of mathematics. We can use it to express in a convenient and precise shorthand the relationships between different objects. Numbers have always been a fascinating and fundamental part of Mathematics. We will use them to provide examples of proofs, algorithms and counter-examples.

Initially we will study the natural numbers $1, 2, 3, \ldots$ and especially Mathematical Induction. Then we expand to consider integers and arithmetic leading to codes like the RSA code used on the internet. Finally we move to rational, real and complex numbers where we lay the logical foundations for analysis. (Analysis is the name given to the study of, for example, the precise meaning of differentiation and integration and the sorts of functions to which these processes can be applied.)

Learning outcomes

By the end of this course, you should:

- understand the need for rigorous proof in mathematics, and be able to apply various different methods, including proof by induction and contradiction, to propositions in set theory and the theory of numbers;
- know the basic properties of the natural numbers, rational numbers and real numbers;
- understand elementary counting arguments and the properties of the binomial coefficients;
- be familiar with elementary number theory and be able to apply your knowledge to the solution of simple problems in modular arithmetic;
- understand the concept of countability and be able to identify typical countable and uncountable sets.

Dynamics and Relativity — 24 lectures, Lent term

This course assumes knowledge from A-level mechanics modules, in particular from the first three modules. If you are not confident that you have the necessary background, you should attend the first lecture of the non-examinable Mechanics course in the Michaelmas term. There is no need to attend the Mechanics course if you are confident of the material in M1, M2 and M3 at A-level (or the equivalent).

This course is the first look at theoretical physics. The course is important not just for the material is contains; it is also important because it serves as a model for the mathematical treatment of all later courses in theoretical physics.

The first 17 lectures are on classical dynamics. The basis of the treatment is the set of laws due to Newton that govern the motion of a particle under the action of forces, and which can be extended to solid bodies. The approach relies heavily on vector methods.

One of the major topics is motion in a gravitational field. This is not only an important application of techniques from this course and the Differential Equations course, it is also of historical interest: it was in order to understand the motions of the planets that Isaac Newton developed calculus.

With the advent of Maxwell's equations in the late nineteenth century came a comfortable feeling that all was well in the world of theoretical physics. This complacency was rudely shaken by Michelson's attempt to measure the velocity of the Earth through the surrounding aether by comparing the speed of light measured in perpendicular directions. The surprising result was that it makes no difference whether one is travelling towards or away from the light source; the velocity of light is always the same. Various physicists suggested a rule of thumb (time dilation and length contraction) which would account for this phenomenon, but it was Einstein who deduced the underlying theory, special relativity, from his considerations of the Maxwell equations.

In this short introduction, the last 7 lectures of this course, there is time only to develop the framework in which the theory can be discussed (the amalgamation of space and time into Minkowski space-time) and tackle simple problems involving the kinematics and dynamics of particles.

Learning outcomes

By the end of this course, you should:

- appreciate the axiomatic nature of, and understand the basic concepts of, Newtonian mechanics;
- be able to apply the theory of Newtonian mechanics to simple problems including the motion of particles, systems of particles and rigid bodies, collisions of particles and rotating frames;
- be able to calculate orbits under a central force and investigate their stability;
- be able to tackle problems in rotating frames;
- be able to solve problems in space-time kinematics and also simple dynamical problems.

3 Course descriptions: non-examinable courses

Mechanics — 8 or more lectures, Michaelmas term

This course is intended to provide background mechanics required for the Dynamics and Relativity course in the Lent term and for later courses in applied mathematics and theoretical physics. It is intended for students who have not done as much mechanics as is contained in three A-level modules. You should attend the first lecture if you are not sure whether you have covered the right material. However, if you have M1, M2 and M3 (or an equivalent), there is no point in attending (unless for revision purposes). The material covered is intended to be useful not only for the course Dynamics and Relativity in the Lent term but also for later courses such as Fluid Dynamics.

Each of the lectures will last about half an hour, and will cover the theory of one important topic (such as conservation of momentum, conservation of energy). It will include a worked example. There will be further examples for you to work through, with solutions provided in video format. It is not intended that the course should entail a significant investment of time. The topic of each lecture will be announced beforehand to enable you to decide whether to attend.

Concepts in Theoretical Physics — 8 lectures, Easter term

This course is intended to provide a mathematical, but not highly detailed, account of eight topics in Theoretical Physics. The style is informal: there is no fixed syllabus and no work to take away. It should be of interest to all students, especially to provide relief from revision.

4 Course descriptions: Part IB Easter term courses

The following courses are lectured in the Easter term and examined in Part IB.

Metric and Topological Spaces — 12 lectures, Easter term

This course may be taken in the Easter term of either the first year or the second year; however, if you are planning to take Complex Analysis (i.e. the course on complex variable theory which has a pure approach; Complex Methods covers roughly the same material with an applied approach), you will find the material in Metric and Topological Spaces very useful.

Continuity is one of the basic ideas developed in Analysis I, and this course shows the value of a very abstract formulation of that idea. It starts with the general notion of distance in the theory of metric spaces and uses that to motivate the definition of topological space. The key topological ideas of connectedness and compactness are introduced and their applications explained. In particular a fresh view emerges of the important result (from Analysis I) that a continuous function on a closed and bounded interval is bounded and attains its bounds.
By the end of this course you should:

Learning outcomes

By the end of this course, you should:

- appreciate the definitions of metric and topological space and be able to distinguish between standard topological and non-topological properties;
- understand the topological notion of connectedness and its relation to path-connectedness;
- understand the topological notion of compactness, know its significance in basic analysis and be able to apply it to identify standard quotients of topological spaces.

Optimisation — 12 lectures, Easter term

This course maybe taken in the Easter term of either the first or the second year.

A typical problem in optimisation is to find the cheapest way of supplying a set of supermarkets from a set of warehouses: in more general terms, the problem is to find the minimum (or maximum) value of a quantity when the variables are subject to certain constraints. Many real-world problems are of this type and the theory discussed in the course are practically extremely important as well as being interesting applications of ideas introduced earlier in Numbers and Sets and Vectors and Matrices.
Topics covered include the simplex algorithm, the theory of two-person games and some algorithms particularly well suited to solving the problem of minimising the cost of flow through a network.

Learning outcomes

By the end of this course, you should:

- understand the nature and importance of convex optimisation;
- be able to apply Lagrangian methods to solve problems involving constraints;
- be able to solve problems in linear programming by methods including the simplex algorithm and duality;
- be able to solve network problems by methods using, for example, the Ford-Fulkerson algorithm and min-cut max-flow theorems.

Variational Principles **12 lectures, Easter term**

This course may be taken in the Easter term of either the first year or the second year; however its content will be very useful to several of the Part IB courses, in particular Methods, Electromagnetism and Quantum Mechanics.
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We all know that the shortest distance between two points in space is a straight line. However, if you are travelling between points on the surface of a sphere, the shortest route is a great circle; this result which can be obtained beautifully using variational principles, i.e. by using mathematical techniques which consider variations across all possible routes to minimise distance. These techniques (the calculus of variations) were developed to solve a deceptively simple problem: what is the route that minimises the transit time between two points for a particle in a gravitational field (which, as every amusement park designer knows, is most definitely not a straight line)? But variational principles have much broader application. Many of the fundamental components of physics (for example, those based around conservation laws) can be derived from variational principles, from Newton's laws of motion, through electromagnetism, quantum mechanics to relativity, in a profoundly elegant way, exploiting the essential interplay of physical concepts with underlying symmetries.

Learning outcomes

By the end of this course, you should:

- understand the concepts of a functional, and of a functional derivative;
- be able to apply constraints to variational problems;
- appreciate the relationship between variational statements, conservation laws and symmetries in physics.

5 Computational Projects

The lectures for this course should be attended in the Easter term of the first year.

The Computational Projects course consists mainly of practical computational projects carried out and written up during the second year. The marks are included in Part IB examination marks. The emphasis is on understanding the physical and mathematical problems being modelled rather than on the details of computer programming. Lectures are given in the Easter term which introduce some of the mathematical and practical aspects of the various projects. More details are available in the Part IB Computational Projects Manual, which is available on-line (`maths.cam.ac.uk/undergrad/catam/IB`). The main programming language is Matlab.

Learning outcomes

By the end of this course, you should:

- be able to programme using a traditional programming language;
- understand the limitations of computers in relation to solving mathematical problems;
- be able to use a computers to solve problems in both pure and applied mathematics involving, for example, solution of ordinary differential equations and manipulation of matrices.