3 Fluid and Solid Mechanics

3.10 Smoke Rings (8 units)

This project discusses a simple model of the motion of smoke rings. Knowledge of Part IB Fluid Dynamics is required and knowledge of the Part II course Classical Dynamics will help with Question 2. The article by Acheson [1] and the book by Saffman [4] may be found helpful.

A smoke ring is a vortex tube wrapped around into a closed circle (a *vortex ring*), which propagates normal to the plane of the circle under its self-induced velocity field. The politically incorrect method of generating them involves the inhalation of noxious substances; a more socially acceptable method involves a volcano [5]. Neglecting various effects, we will throughout this project crudely model a three-dimensional axisymmetric vortex ring of diameter a and strength κ by a pair of point vortices in two-dimensional fluid of strengths κ and $-\kappa$, a distance a apart.

1 2D vortex dynamics: Theory

Question 1 Show that the equations of motion of a set of point vortices of strengths κ_i at positions $(x_i(t), y_i(t))$, in a two-dimensional inviscid fluid which is otherwise at rest, are

$$\begin{aligned}\frac{\mathrm{d}x_i}{\mathrm{d}t} &= -\frac{1}{2\pi}\sum_{j\neq i}\frac{\kappa_j(y_i-y_j)}{r_{ij}^2}\\ \frac{\mathrm{d}y_i}{\mathrm{d}t} &= \frac{1}{2\pi}\sum_{j\neq i}\frac{\kappa_j(x_i-x_j)}{r_{ij}^2},\end{aligned} \tag{1}$$

where

$$r_{ij} = ((x_i-x_j)^2+(y_i-y_j)^2)^{1/2}. \tag{2}$$

Question 2 Show carefully that the equations of motion can be written in the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \kappa_i^{-1}\frac{\partial H}{\partial y_i}, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\kappa_i^{-1}\frac{\partial H}{\partial x_i} \qquad \text{(no summation)} \tag{3}$$

where

$$H = -\frac{1}{4\pi}\sum_{\substack{i,j\\ i\neq j}}\kappa_i\kappa_j\log r_{ij}. \tag{4}$$

H is invariant under space translations and rotations, which implies the existence of three scalar conserved quantities. What are they?

Hint: Suppose H has a continuous family of symmetries, i.e.

$$H(X(x,y,\delta), Y(x,y,\delta)) = H(x,y)\,,$$

such that $X(x,y,0)=x$ and $Y(x,y,0)=y$. Observe that

$$0 = \sum_i\left(\frac{\partial H}{\partial x_i}\frac{\partial X_i}{\partial \delta}+\frac{\partial H}{\partial y_i}\frac{\partial Y_i}{\partial \delta}\right)_{\delta=0}.$$

H is also invariant under time translations — what conserved quantity does this give?

Programming Task: Write a program to integrate the equations of motion (1). You should use an adaptive stepsize ODE integrator. You will find it useful to write your code to handle arbitrarily many vortices.

2 Simulations of smoke rings

Question 3 Use your code to investigate the motion of a single "axisymmetric vortex ring" under this model. This problem can also be solved analytically; use the analytic solution to test your code. Also demonstrate that your code preserves the constants of the motion.

Question 4 What happens when two smoke rings are fired towards each other on the same axis? Describe the resulting motion, giving clear physical explanations for the behaviour. You should start by considering two rings with equal strengths and widths, but should also explain what happens in the general case.

Question 5 What happens when two smoke rings are fired in the same direction on the same axis? Describe the resulting motion, giving clear physical explanations for the behaviour. You should start by considering two rings with equal strengths and widths, but should also explain what happens in the general case. You should not need to integrate to large times, but you should (where relevant) show several cycles of the motion.

Question 6 We have made a number of modelling assumptions in reducing the full three-dimensional problem to this simple two-dimensional version. How good are they? Would the behaviour that you have observed in question 5 occur in a real physical system? Are the physical explanations that you gave in questions 4 and 5 for two dimensions relevant in the real geometry? What other possible effects have we neglected?

3 Symmetries and instabilities

Question 7 Repeat a typical one of the simulations you did for question 5, but now integrate to large times and show your output. What happens? Is the resulting behaviour physically plausible for a pair of 3D vortex rings? What happens if you tighten the error tolerances of your ODE solver?

Programming Task: Produce a program which can only model coaxial smoke rings, but which explicitly enforces the symmetry about the axis. In other words, use the mirror symmetry of the model system to reduce the number of ODEs that you have to solve.

Note that this is not the right way to handle symmetries and conserved quantities in the numerical solution of ODEs. See Iserles et. al. *[2] for more information. For this project, however, this method will suffice.*

Question 8 Use this new program to repeat the simulation you did for question 7. Show your output and comment.

4 More smoke rings

Question 9 Consider three coaxial smoke rings, fired in the same direction. Use your new program to investigate the resulting motion. Give a survey of the possible kinds of behaviour, including a selection of your plots (4 should suffice).

Note that the parameter space you have to search is rather large. You should think of ways to reduce it.

References

[1] Acheson, D. J. 2000, Instability of vortex leapfrogging, *Eur. J. Phys.*, **21**, 269–273.

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[2] Iserles, A., Munthe-Kaas, H. Z., Nørsett, S. P. and Zanna, A. 2000, Lie-group methods, *Acta Numerica*, **9**, 215–365.

[3] Pullin, D. I. and Saffman, P. G. 1991, Long-time symplectic integration: the example of four-vortex motion, *Proc. Roy. Soc. Lond. A*, **432**, 481–494.

[4] Saffman, P. G. 1992, *Vortex Dynamics*, CUP.

[5] `http://www.swisseduc.ch/stromboli/etna/etna00/etna0002photovideo-en.html`

* At the time of writing http://iopscience.iop.org/0143-0807/21/3/310 should take you there directly.

