2 Waves

2.2 Dispersion (7 units)

This project assumes only the elementary properties of dispersive waves, covered in the Part II Waves course (but the relevant material can be found in the references).

1 Introduction

This project illustrates the way in which a disturbance in a 'dispersive-wave' system can change shape as it travels. In order to fix ideas we shall consider one-dimensional waves, depending on a single spatial coordinate x and time t, which are modelled by a system of linear constant-coefficient partial differential equations that is (i) second-order in time and (ii) time-reversible. Such a system has single-Fourier-mode (aka 'plane-harmonic-wave') solutions proportional to

$$\mathrm{e}^{ikx \mp i\omega(k)t} \tag{1}$$

for any real '[angular] wavenumber' k, where the '[angular] frequency' ω is real and related to k by a system-dependent 'dispersion relation'. The waves are 'dispersive' if ω is not directly proportional to k (and so 'group velocity' $d\omega/dk$ and 'phase velocity' ω/k vary with k, and are unequal). As an example, one-dimensional 'capillary-gravity' waves on the free surface of incompressible fluid of uniform depth h have dispersion relation

$$\omega^2 = \left(gk + \rho^{-1}\gamma k^3\right)\tanh(kh) \tag{2}$$

where g is gravitational acceleration, ρ the fluid density and γ the coefficient of surface tension.

If the disturbance is described by a function $F(x,t)$, representing say the [non-dimensionalised] vertical displacement of the fluid surface, the general solution for F will be a superposition of all Fourier modes of the form (1):

$$F(x,t) = \int_{-\infty}^{\infty} \left(a_+(k)\,\mathrm{e}^{ikx - i\omega(k)t} + a_-(k)\,\mathrm{e}^{ikx + i\omega(k)t}\right) dk \ , \tag{3}$$

where the amplitudes $a_+(k)$ and $a_-(k)$ are fixed by the initial conditions. For simplicity we shall take these to be

$$F(x,0) = \exp\left(-\frac{x^2}{\sigma^2}\right)\cos(k_0 x) \quad \text{and} \quad \frac{\partial F}{\partial t}(x,0) = 0 \ . \tag{4}$$

where σ and k_0 are constants.

Question 1 Show that (3) then can be written as

$$F(x,t) = \int_{-\infty}^{\infty} A(k)\cos[\omega(k)t]\,\mathrm{e}^{ikx}\,dk \ , \tag{5}$$

where $A(k)$ is a real function to be determined.

In order to plot the solution some method is needed for evaluating the Fourier integral (5).

2 The Discrete Fourier Transform

The Fourier Transform $\hat{G}(k)$ of a function $G(x)$ may be defined by*

$$\hat{G}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x)\,\mathrm{e}^{-ikx}\,dx \ , \tag{6}$$

with inverse

$$G(x) = \int_{-\infty}^{\infty} \hat{G}(k)\,\mathrm{e}^{ikx}dk \ . \tag{7}$$

The integral (7) can be approximated by the discretisation

$$\Delta k \sum_{n=-N/2+1}^{N/2} \hat{G}_n \mathrm{e}^{in(\Delta k)x} \ , \quad \hat{G}_n = \hat{G}(n\Delta k) \tag{8}$$

provided that Δk is small enough to resolve the variation of the integrand with k, and that $\hat{G}(k)$ is only significant for $|k| < \frac{1}{2}N\Delta k$. With $\Delta k = 2\pi/L$ and $\Delta x = L/N$, this approximates $G(x = m\Delta x)$ by

$$g_m \equiv \frac{2\pi}{L} \sum_{n=-N/2+1}^{N/2} \hat{G}_n \mathrm{e}^{2\pi imn/N} \quad \text{for } -N/2+1 \leqslant m \leqslant N/2 \ . \tag{9}$$

[note that g_m is periodic in m with period N, and cannot be expected to give a useful approximation to $G(x = m\Delta x)$ for $|m| > N/2$, i.e. for $|x| > L/2$, since the e^{ikx}-factor in the integrand would be chronically under-resolved].

(9) is the *exact* inverse of

$$\hat{G}_n = \frac{L}{2\pi N} \sum_{m=-N/2+1}^{N/2} g_m \mathrm{e}^{-2\pi imn/N} \quad \text{for } -N/2+1 \leqslant n \leqslant N/2 \ ; \tag{10}$$

the right-hand side is a discretisation of the integral (6) with $k = n\Delta k$, but that will not required in this project. The so-called *Discrete Fourier Transform* (10) and its inverse (9) converge to the Fourier Transform (6) and its inverse (7) in the double limit $L \to \infty$, $N/L \to \infty$.

3 The Fast Fourier Transform

The Fast Fourier Transform (FFT) technique is a quick method of evaluating sums of the form

$$\lambda_m = \sum_{n=0}^{N-1} \mu_n \, (\zeta_N)^{smn} \ , \quad m = 0, \ldots, N-1 \ , \quad \zeta_N = \mathrm{e}^{2\pi i/N} \ , \quad s = \pm 1 \tag{11}$$

where the μ_n are a known sequence, and N is a product of small primes, preferably a power of 2. A brief outline of the FFT is given in the appendix for reference, but it is not necessary to understand the details of the algorithm in order to complete the project – indeed, you are strongly advised to use a black-box FFT procedure such as Matlab's `fft/ifft`. Note that since

$$(\zeta_N)^{smn} = (\zeta_N)^{s(m\pm N)n} = (\zeta_N)^{sm(n\pm N)} \tag{12}$$

*There are various conventions regarding the sign of the exponent and the placement of the 2π-factor.

the sums in (9) and (10) can be converted to the form (11) by repositioning part of the series (and Matlab arrays are indexed from 1 to N rather than 0 to $N-1$). Similar considerations also apply to available routines in other languages, and you may also need to take special care regarding sign conventions and scaling.

Programming Task: Write a program to compute a DFT approximation to $F(x,t)$.

4 No Dispersion

In the limit of 'shallow water' ($|k|h \ll 1 \Rightarrow \tanh(kh) \approx kh$) and negligible surface tension ($\rho^{-1}\gamma|k|^3 \ll g|k|$), the dispersion relation (2) can be approximated by the 'dispersionless'

$$\omega^2 = c_0^2 k^2 \tag{13}$$

with $c_0 = \sqrt{gh}$. The integral (5) can then be evaluated analytically.

Question 2

Use this to test the program for t up to 10 s, taking $\sigma = 0.5\,\mathrm{m}$, $k_0 = 0\,\mathrm{m}^{-1}$ and $c_0 = 1\,\mathrm{m\,s}^{-1}$ [so $h \approx 0.1\,\mathrm{m}$ if $g = 9.81\,\mathrm{m\,s}^{-2}$]. Choose appropriate values for the parameters L and N so that your plots are correct to 'graphical accuracy'; present evidence of this accuracy in your write-up. Comment on your results [e.g. on the appropriateness of the 'shallow-water' approximation for these parameter values].

5 Gravity Waves

The 'deep-water' ($|k|h \gg 1 \Rightarrow \tanh(kh) \approx \mathrm{sign}(k)$) and negligible-surface-tension limit of the dispersion relation (2) is

$$\omega^2 = g|k|. \tag{14}$$

Question 3 Take $g = 9.81\,\mathrm{m\,s}^{-2}$ and in the first instance use initial condition (4) with $\sigma = 1$ m, $k_0 = 0\,\mathrm{m}^{-1}$.

- For $t = 2\,\mathrm{s}$ investigate the effects of changing the values of L and N (maybe start with $L = 32$ m and $N = 32$). Report the results of this investigation in your write-up, especially with regard to the errors in the solution, using both numerical values and plots.
- Display graphical results to illustrate how the solution for this initial condition evolves for t up to at least 6 s, giving justification for your choices of L and N. Do likewise for the initial condition (4) with $\sigma = 6$ m and $k_0 = 1\,\mathrm{m}^{-1}$, for t up to at least 20 s.
- Comment on the solutions, particularly in the light of group and phase velocity.

6 Capillary Waves

Consider now the dispersion relation for 'deep-water' surface waves when surface-tension effects dominate over gravitational:

$$\omega^2 = \rho^{-1}\gamma|k|^3 \ . \tag{15}$$

Question 4 Perform similar calculations to those in Q3 for water with $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ and $\gamma = 0.074\,\mathrm{kg\,s^{-2}}$, using the initial condition (4) with $\sigma = 0.002$ m, $k_0 = 0\,\mathrm{m^{-1}}$ and with $\sigma = 0.005$ m, $k_0 = 1250\,\mathrm{m^{-1}}$, for t up to at least 0.1 s.

- Compare and contrast your results with those in Q3. You will want to use different value(s) for L (and maybe N): can the concept of group velocity help in choosing a suitable L for given time?
- How much difference would it make to these results if the exact 'deep-water' dispersion relation
$$\omega^2 = g|k| + \rho^{-1}\gamma|k|^3 \tag{16}$$
were used, with $g = 9.81\mathrm{m\,s^{-2}}$?

References

Billingham, J. & King, A. C., *Wave Motion: Theory and Applications*, CUP.

Lighthill, M. J., *Waves in Fluids*, CUP.

Whitham, G. B., *Linear and Nonlinear Waves*, Wiley.

Appendix: The Fast Fourier Transform

For simplicity restrict to the optimal case $N = 2^M$. Then the DFT (11) can be split into its even and odd terms

$$\lambda_m = \underbrace{\sum_{n'=0}^{N/2-1} \mu_{2n'} \left(\zeta_{N/2}\right)^{smn'}}_{\lambda_m^E} + \left(\zeta_N\right)^{sm} \underbrace{\sum_{n'=0}^{N/2-1} \mu_{2n'+1} \left(\zeta_{N/2}\right)^{smn'}}_{\lambda_m^O} \tag{17}$$

and since λ_m^E and λ_m^O are periodic in m with period $N/2$, and $\left(\zeta_N\right)^{sN/2} = -1$,

$$\lambda_{m+N/2} = \lambda_m^E - \left(\zeta_N\right)^{sm} \lambda_m^O \ . \tag{18}$$

Thus if the half-length transforms λ_m^E, λ_m^O $(0 \leqslant m \leqslant N/2-1)$ are known, it 'costs' $\frac{1}{2}N$ products to evaluate the λ_m for $0 \leqslant m \leqslant N-1$. The process can be performed recursively M times, giving a decomposition in terms of N transforms of length one – which are just the original μ_n $(0 \leqslant n \leqslant N-1)$.

To execute an FFT, start with these length-one transforms; at the s-th stage, $s = 1, 2, \ldots, M$, assemble 2^{M-s} transform of length 2^s from transforms of length 2^{s-1}, at a 'cost' of $2^{M-1} = \frac{1}{2}N$ products. The complete DFT is formed after M stages, i.e. after $\frac{1}{2}N\log_2 N$ products, as opposed to N^2 products in naive matrix multiplication – so for $N = 1024 = 2^{10}$ the cost is 5×10^3 products compared to 10^6 products!

For more details, see for example Press *et al.*, *Numerical Recipes*, CUP.

