23 Astrophysics

23.8 Chaos in the Encounter Map (8 units)

No knowledge of astrophysics is assumed or required for this project. All relevant equations are defined and explained in the questions.

The orbit of a body around the Sun is described by an ellipse:

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos(\theta - \varpi)}, \tag{1}$$

where r is the radial position, a is the *semi-major axis*, e is the *eccentricity*, θ is the *true longitude*, and ϖ is the *longitude of perihelion*, where perihelion is the distance of closest approach to the Sun. It is sometimes useful to combine e and ϖ into a single complex variable $z = e\exp(\mathrm{i}\varpi)$, where $\mathrm{i} = \sqrt{-1}$. It is also useful to define the *mean longitude* which increases linearly with time $\lambda = nt$, where n is the *mean motion* of the body given by Kepler's third law $n^2 = Gm_\odot/a^3$, G is Newton's gravitational constant and $m_\odot$ is the mass of the Sun.

No such analytical solution exists for the three-body problem. This project investigates an algebraic mapping called the *encounter map* which considers a massless test particle orbiting in the gravitational potential of two massive bodies: a star of Solar mass (i.e. $m = 1m_\odot$) and a planet on a circular orbit with mass ratio $\mu = m_\mathrm{pl}/m_\odot \ll 1$. The encounter map considers the particle to orbit the Sun on a Keplerian orbit defined by equation 1 until the particle reaches conjunction with the planet, i.e., when they are at the same mean longitude, at which point the orbital elements are changed impulsively and the particle continues on its new orbit until the next conjunction. Note, that in the following, the planet's properties are denoted with a subscript $_\mathrm{pl}$ and the test particle's properties are unsubscripted or subscripted by a number to keep track of conjunctions.

A derivation of the encounter map requires working in a reference frame corotating with the planet and centered on it, with the x-axis pointing away from the star and the y-axis pointing in the direction of the planet's motion. The motion of the particle is governed by *Hill's Equations:*

$$\begin{aligned}
\ddot{x} - 2n_\mathrm{pl}\dot{y} - 3n_\mathrm{pl}^2 x &= -\frac{Gm_\mathrm{pl}x}{\Delta^3} \\
\ddot{y} + 2n_\mathrm{pl}\dot{x} &= -\frac{Gm_\mathrm{pl}y}{\Delta^3},
\end{aligned} \tag{2}$$

where $\Delta = \sqrt{x^2 + y^2}$ is the distance from the planet to the particle. A conserved quantity of these equations is the *Jacobi constant:*

$$C_\mathrm{H} = 3n_\mathrm{pl}^2 x^2 + \frac{2Gm_\mathrm{pl}}{\Delta} - \dot{x}^2 - \dot{y}^2. \tag{3}$$

Question 1 Consider motion in Hill's equations before or after a conjunction. Derive the solution of Hill's Equations:

$$\begin{aligned}
x &\sim D_1\cos n_\mathrm{pl}t + D_2\sin n_\mathrm{pl}t + D_3, \\
y &\sim -2D_1\sin n_\mathrm{pl}t + 2D_2\cos n_\mathrm{pl}t - \frac{3}{2}D_3 n_\mathrm{pl}t + D_4,
\end{aligned} \tag{4}$$

in the limit $\Delta \to \infty$, where D_i are constants.

It can be shown that, for elliptic orbits, $D_1 + iD_2 = -a_{\mathrm{pl}} z \exp(-i\lambda_{\mathrm{c}})$ and $D_3 = a - a_{\mathrm{pl}}$, where λ_{c} is the longitude of the planet at conjunction. It can also be shown that, as a result of a single conjunction, to lowest order in mass ratio, the coefficients D_1 and D_3 are unchanged, but that

$$\Delta D_2 = -g \, \mathrm{sign}(D_3) \, \frac{\mu a_{\mathrm{pl}}^3}{D_3^2} \tag{5}$$

where $g = 2.239566674$. Note that this result requires that the test particle has a low-eccentricity orbit which is not too distant from the planet, so that $e \ll 1$ and $\epsilon = (a - a_{\mathrm{pl}})/a_{\mathrm{pl}} \ll 1$, but the particle must also be sufficiently far from the planet that it feels only a small perturbation at conjunction, requiring $\epsilon > e + (\mu/3)^{1/3}$ so that the particle's eccentricity does not carry it into the region in which the planet's gravity is stronger than the star's.

Question 2 Using the conserved quantity C_{H}, find a more accurate expression for the change in D_3, assuming that D_1 remains unchanged, and that ΔD_2 is given by equation 5. Hence derive the mapping from the orbital elements prior to the n-th conjunction, occurring at longitude λ_n, z_n and ϵ_n, to those at the $(n+1)$-th conjunction at λ_{n+1}:

$$z_{n+1} = z_n + \frac{ig \exp i\lambda_n}{\epsilon_n^2} \mathrm{sign}(\epsilon_n)\mu \tag{6}$$

$$\epsilon_{n+1} = \epsilon_n \sqrt{1 + \frac{4(|z_{n+1}|^2 - |z_n|^2)}{3\epsilon_n^2}} \tag{7}$$

$$\lambda_{n+1} = \lambda_n + 2\pi |(1 + \epsilon_{n+1})^{-3/2} - 1|^{-1}. \tag{8}$$

Question 3 To make the map area-preserving, it is necessary to change instances of ϵ_n to ϵ_1 in the equation for z_{n+1}. With this change, write a program to implement the map derived in Question 2. Consider a planet at $1\,\mathrm{AU}$ [$1\,\mathrm{AU} = 1$ Astronomical Unit, the mean distance of the Earth from the Sun] with a mass of 10^{-5} times solar, orbiting a solar-mass star. Calculate the Jacobi constant C_{H} for a particle initially at $1.08\,\mathrm{AU}$ on a circular orbit. Generate a population of 100 other particles with the same Jacobi constant, but with eccentricities up to 0.07, and evolve their orbits for 1000 iterations of the map. Show some representative examples of the evolution of orbital elements. Are your conclusions about the particles' behaviour affected if more iterations are performed?

Question 4 Let $y_n = z_n \exp(-i\theta_n)$, where $\theta_n = \lambda_n - \pi \left|(1 + \epsilon_n)^{-3/2} - 1\right|^{-1} + \pi$. Plot, on a single figure, the evolution of y in the Argand plane for the trajectories calculated in Question 3, and describe the behaviour (this is called a *Poincaré surface of section*).

A *mean motion resonance* occurs when the ratio of the bodies' mean motions is an integer commensurability $j : j - k$ such as 3:2. The strongest and most important are the first-order resonances where the ratio is $j : j - 1$. A particle is in resonance with a planet if the relevant *resonant argument* is librating rather than circulating. The resonant argument is given by

$$\phi = j\lambda - (j - k)\lambda_{\mathrm{pl}} - k\varpi \tag{9}$$

for a $j : j - k$ resonance; typically the resonant argument librates about π. The resonances nominally occur at specific semi-major axes, but have a finite width, so that a particle at $1.6\,\mathrm{AU}$ may still be in the 2:1 resonance with a planet at $1\,\mathrm{AU}$. The width of the first order resonances depends on eccentricity, but is given roughly by

$$\delta a/a \approx \mu^{2/3} j^{1/3} \tag{10}$$

for large j.

Question 5   For the parameters of question 3, calculate the eccentricies for particles located in first-order resonances. Give the physical interpretation of the angle θ, and relate the argument of the complex variable y_n to the resonant argument ϕ for a given resonance. Hence determine whether these resonances explain any features of the Poincaré surface of section.

Question 6   Devise a method to automatically classify the trajectories as regular or chaotic. This distinction does not need to be rigorously demonstrated, but can be based on your observations of the behaviour. Describe the method and what motivates it. Check that it agrees with a classification by eye of the trajectories calculated in previous questions. (*Hint:* You may find it useful to look at the Fourier transform of the time-series of orbital elements).

Question 7   Now consider particles on initially circular orbits. For a range of mass ratios $\mu \in [10^{-9}, 10^{-3}]$, find whether initially circular orbits are chaotic for $\epsilon \in [10^{-2.5}, 10^{-0.5}]$, and produce an image showing where in this parameter space the trajectories are chaotic. Find, for each μ, the location of the innermost regular orbit ϵ_{crit}. Plot ϵ_{crit} as a function of μ, and find the best power-law fit, $\epsilon_{\mathrm{crit}} = \alpha \mu^\beta$.

Question 8   Using Equation (10), find the critical semi-major axis at which first order resonances overlap.