14 General Relativity

14.5 Cosmological distances (8 units)

Although this project is based on general relativistic cosmology, no detailed knowledge of General Relativity is required. All relevant equations are defined and explained in the project itself.

1 Introduction

In cosmology there are many ways to specify the distance between two points because, in the expanding Universe, the distances between objects are changing and Earth-bound observers look back in time as they look out in distance. All these distances measure the separation between events on radial null trajectories, trajectories of photons which terminate at the observer.

The metric for a homogeneous isotropic universe, in spherical polar coordinates for the spatial part, is

$$ds^2 = c^2dt^2 - R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{1}$$

where, by a suitable choice of radial coordinate r, $k = -1$, 0 or 1 for open, Euclidean or closed geometries.

In this metric the redshift relative to an observer at the spatial origin is given by

$$1 + z = R(t_0)/R(t_1), \tag{2}$$

where t_0 is the coordinate time at which the photon is received and t_1 that at which it was emitted. Thus, for a given observer, the redshift depends only on the radial scale factor of the Universe at the time the photon was emitted divided by its value at the observer's time. The redshift is important because it can be measured easily from the observed wavelengths of atomic transition lines with known rest wavelengths.

When the matter density at time t is ρ and the pressure is zero one of the Einstein field equations with the cosmological term becomes

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho, \tag{3}$$

where $\dot{R} = \frac{dR}{dt}$ and Λ is a constant. The other field equation can be combined with this to give the conservation of matter equation

$$\rho R^3 = \text{const.} \tag{4}$$

For small distances the redshift $cz = H_0 d$, where d is the distance to the source. Then H_0, the Hubble constant, gives the local expansion rate. It is often written in the form $H_0 = 100h\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} = 3.2409 \times 10^{-18} h\,\text{s}^{-1}$, where h is dimensionless. The actual value of h is still uncertain, and hotly debated, but most would agree on measurements of 0.72 ± 0.08. The megaparsec is an astronomical length unit appropriate for separations between galaxies, $1\,\text{Mpc} = 3.0856 \times 10^{22}\,\text{m}$. The Hubble time $t_\text{H} = 1/H_0 = 3.0856 \times 10^{17} h^{-1}\,\text{s}$ and the Hubble distance $D_\text{H} = c/H_0 = 9.26 \times 10^{25} h^{-1}\,\text{m}$. Take the number of seconds in one year to be 3.1556926×10^7 s.

Our Universe can be described by two parameters, the matter density now ρ_0 and the cosmological constant Λ, and we can express these in a dimensionless form using H_0 as

$$\Omega_\text{m} \equiv \frac{8\pi G\rho_0}{3H_0^2} \tag{5}$$

and

$$\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_0^2}. \tag{6}$$

By means of equation (3), at time t_0, the curvature value k can be parameterised by Ω_k so that

$$\Omega_\mathrm{m} + \Omega_\Lambda + \Omega_k = 1. \tag{7}$$

Then the function

$$E(z) = \sqrt{\Omega_\mathrm{m}(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$$

is proportional to the time derivative of the logarithm of the scale factor, $\dot{R}/R$, at redshift z (see *e.g.* Peebles 1993, pp 310 – 321).

Where specific values are required in what follows you should take $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

2 Lookback Time

The lookback time t_L is the difference between the age t_0 of the Universe now and the age t_e when the photons were emitted

$$t_\mathrm{L} = t_\mathrm{H} \int_0^z \frac{dz'}{(1+z')E(z')}. \tag{8}$$

Question 1 If $\Omega_\mathrm{m} = 1$ and $\Omega_\Lambda = 0$, obtain an expression for the lookback time to an object with redshift z and show that the age of the Universe is $t_\mathrm{L}(z = \infty) = \frac{2}{3}t_\mathrm{H}$.

Question 2 Write a program to determine the lookback time in Gyr for general H_0, Ω_m and Ω_Λ. If $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, tabulate the lookback time to $z = 0.1$, 1.0, 2.0, 4.0 and 6.7 (one of the highest individual object redshifts measured so far) for

(1) an Einstein-de-Sitter universe $\Omega_\mathrm{m} = 1$, $\Omega_\Lambda = 0$,

(2) a classical closed universe $\Omega_\mathrm{m} = 2$, $\Omega_\Lambda = 0$,

(3) a baryon dominated low density universe $\Omega_\mathrm{m} = 0.04$, $\Omega_\Lambda = 0$ and

(4) the currently popular Universe $\Omega_\mathrm{m} = 0.27$, $\Omega_\Lambda = 0.73$.

What is the age of the Universe for each of these models [to the nearest 100 million years]?

Produce a graph showing lookback time against redshift for the four models and comment on any overall trends.

3 Distance Measures

There are three useful ways to define distance.

(1) The line of sight comoving distance

$$D_\mathrm{C} = D_\mathrm{H} \int_0^z \frac{dz'}{E(z')}. \tag{9}$$

(2) The angular diameter distance is the ratio of an object's physical size to its angular size (in radians). For an object of size ℓ at redshift z the angular size is $\theta = \ell/D_\mathrm{A}$, where θ is a small angle (so $\sin\theta \approx \tan\theta \approx \theta$).

$$D_A = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}(1+z)} \sinh\left[\sqrt{\Omega_k} D_C/D_H\right], & \text{for } \Omega_k > 0, \\ D_C/(1+z), & \text{for } \Omega_k = 0, \\ D_H \frac{1}{\sqrt{|\Omega_k|}(1+z)} \sin\left[\sqrt{|\Omega_k|} D_C/D_H\right], & \text{for } \Omega_k < 0. \end{cases} \tag{10}$$

(3) The luminosity distance D_L is defined by the relationship between the observed photon energy flux f, integrated over all frequencies, and the intrinsic energy output from the source L by

$$f = \frac{L}{4\pi D_L^2}.$$

It is related to the angular diameter distance by

$$D_L = (1+z)^2 D_A. \tag{11}$$

Question 3 Obtain an analytic expression for the angular diameter distance D_A as a function of redshift in the case where $\Omega_m = 1$ and $\Omega_\Lambda = 0$ and show that it has a maximum value when $z = 1.25$.

Question 4 Write a program to determine the luminosity and angular diameter distances given the redshift z and plot the dimensionless values D_A/D_H and D_L/D_H for redshifts $0 < z < 7$ for $(\Omega_m, \Omega_\Lambda) = (1, 0)$, $(0.04, 0)$ and $(0.27, 0.73)$. For these three cases tabulate the values at redshifts $z = 1$, 1.25, 2.0 and 4.0.

4 Comoving Volume

The comoving volume V_C is the volume measure in which the number density of non-evolving objects is constant with redshift. The comoving volume element in solid angle $\sin\theta d\theta d\phi$ and redshift interval dz is

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} \sin\theta \, dz \, d\theta \, d\phi.$$

Integrating this from the present to redshift z gives the total comoving volume over the whole sky to redshift z,

$$V = \frac{4\pi}{3} \frac{D_L^3}{(1+z)^3} = \frac{4\pi}{3} D_C^3 \qquad \text{for} \quad \Omega_k = 0. \tag{12}$$

A method to test whether a sample of objects has a uniform comoving density and luminosity which does not change with cosmic time is to use the $< V/V_{max} >$ test. It is assumed that all objects with observed flux $f > f_0$ are detected and the observed flux f and the redshift z are measured for each object. For a given luminosity L there is a maximum redshift $z_{max}(L)$ at which the observed flux is f_0 so that the object is just included. Corresponding to this redshift is a maximum volume $V_{max}(L)$. Then, if we have a distribution of luminosities so that $\Phi(L)dL$ is the number per unit comoving volume with luminosity between L and $L + dL$, the total number of objects in the sample is

$$\int_0^\infty \Phi(L) \int_0^{V_{max}(L)} dV dL.$$

where V is the comoving volume.

Question 5 Show that for a uniform comoving distribution of objects the expectation value $\langle V/V_{max} \rangle = \frac{1}{2}$.

Question 6 Write a program to read pairs of numbers z and f/f_0, determine V and $V_{\max}$ for these for Universe models for which $\Omega_k = 0$ and determine the average value $< V/V_{\max} >$ for each model.

Verify that for small values of z the program gives the Euclidean limit, for individual cases $V/V_{\max} \propto (f/f_0)^{-\frac{3}{2}}$.

Apply the program to the sample, listed below, of 114 quasars from an area of sky. What is the value of $< V/V_{\max} >$ for this sample if $\Omega_m = 0.27$ and $\Omega_\Lambda = 0.73$? Is the value of $< V/V_{\max} >$ what you would expect from a constant comoving population? How might you interpret the result you obtain?

Question 7 The sample in the previous question was also subject to the constraints $z > 0.20$ and $z < 3.0$, because it is only in this range that an object be recognised as a quasar. How would you modify the $V/V_{\max}$ quantity so that for a uniform distribution in this redshift range the average value is still $\frac{1}{2}$? What is the result of using this on the sample of 114 quasars?

References

[1] Peebles, P. J. E., 1993, *Principles of Physical Cosmology*, Princeton University Press.

Quasar data. The following may also be found in the file `quasar.dat` in the `data` directory on the CATAM website:

z	f/f_0	z	f/f_0	z	f/f_0	z	f/f_0	z	f/f_0	z	f/f_0
0.202	1.570	0.217	3.250	0.225	2.884	0.237	3.630	0.246	1.213	0.259	1.330
0.274	1.614	0.298	1.330	0.315	2.032	0.322	1.066	0.332	1.976	0.351	1.018
0.362	1.096	0.373	1.191	0.385	2.937	0.402	2.355	0.433	1.853	0.449	4.168
0.460	5.105	0.479	1.706	0.492	1.629	0.507	1.940	0.530	1.472	0.549	1.419
0.571	2.511	0.582	2.089	0.590	1.599	0.609	1.406	0.624	1.018	0.641	1.018
0.659	3.564	0.672	2.511	0.679	2.013	0.692	1.294	0.714	1.294	0.723	1.584
0.737	1.342	0.754	1.076	0.774	1.753	0.781	2.128	0.791	2.779	0.803	2.421
0.832	1.106	0.847	1.527	0.874	1.158	0.892	1.202	0.913	2.167	0.934	1.629
0.955	1.887	0.973	2.208	0.993	2.558	1.012	1.355	1.025	1.247	1.040	1.318
1.056	1.213	1.072	1.803	1.092	1.330	1.115	1.342	1.140	1.086	1.152	1.180
1.182	1.180	1.205	1.393	1.220	1.247	1.234	1.342	1.247	2.535	1.263	1.047
1.288	1.541	1.313	1.028	1.332	1.037	1.343	1.235	1.376	2.779	1.388	1.202
1.400	1.086	1.440	1.127	1.455	1.009	1.469	1.056	1.487	1.330	1.511	1.330
1.543	1.028	1.559	1.202	1.583	1.819	1.593	1.294	1.619	1.614	1.641	3.047
1.664	1.393	1.684	1.158	1.700	1.513	1.727	1.629	1.756	1.137	1.776	1.355
1.810	2.831	1.844	1.018	1.878	2.208	1.913	2.606	1.941	1.342	1.961	1.555
1.976	2.910	2.005	1.106	2.035	1.306	2.075	1.887	2.092	1.958	2.106	1.367
2.134	1.406	2.187	2.089	2.244	1.202	2.297	1.106	2.329	1.009	2.388	1.737
2.442	1.644	2.523	1.000	2.595	1.393	2.649	1.282	2.786	1.527	2.936	1.445