14 General Relativity

14.1 Particle or Photon Orbits near a Black Hole (8 units)

This project is most suitable for those attending the Part II course General Relativity.

The aim of this project is to investigate the motion of a particle or photon in the Schwarzschild solution to the Einstein field equations.

1 Theory

Use the Lagrangian

$$L = F(r)\dot{t}^2 - \frac{\dot{r}^2}{F(r)} - (r\dot{\phi})^2 \ ,$$

where $F(r) = 1 - 1/r$, and dots denote differentiation with respect to a variable, say s, which will be the affine parameter along the geodesic. (The units are such that $c = 1$ and $2GM = 1$ which makes the Schwarzschild radius $r = 1$.)

Question 1 Using Lagrange's equations and the Lagrangian above, obtain the equations of motion for a particle and a photon and establish equations for orbits in (u, ϕ) space, where $u = 1/r$, using the second derivative of u with respect to ϕ. You should set $L = 0$ or 1 depending on whether you are using photons or particles, and use this in the differential equation if you need it.

2 Particle Orbits

To address the questions below you should integrate the equations using one of the Runge–Kutta routines or another of your choice.

Question 2 A particle is at a distance corresponding to $r = 6$, with $dr/d\phi = 0$ initially. For a range of angular momentum parameters from 0 to that for a circular orbit, determine

- the orbit in (r, ϕ) space, giving graphical output of the orbits in this space;
- for what range of angular momentum parameters the particle is captured by the black hole (i.e., it reaches $r = 1$);
- the proper time taken for a particle to fall into a black hole if it does so.

Question 3 What is the nature of the orbits if $r = 2.5$ initially?

Question 4 Compare your computed results with the theoretical results for some cases where analytic solutions are possible. In particular you should examine the stability of circular particle orbits at small radial distances from the black hole, and compare this with the numerical results.

Question 5 How would the outputs change if the problem was formulated using the second derivative of r with respect to t, and the first derivative of ϕ with respect to t?

3 Particle Scattering

Consider a particle far from the hole with speed v and impact parameter b (the *impact parameter* is the minimum distance between the centre of the hole and the undeflected path that the particle would follow in the absence of the hole). For given v there is a critical value of b, say b_{crit}, such that particles with $b < b_{crit}$ are captured, while particles with $b > b_{crit}$ are not. The *cross-section* $\sigma(v)$ is defined by $\sigma(v) = \pi b_{crit}^2$.

> **Question 6** Devise a program to obtain $\sigma(v)$ and to produce a graph of $\sigma(v)$ for $0 < v < 1$. Comment on the limits $v \ll 1$, $v \approx 1$.

4 Photon scattering

> **Question 7** A photon far from the hole is directed towards it with impact parameter b. Compute the deflection angles when b is large, and compare this with results obtained analytically on the assumption that the bend angle is small.
>
> Compute the angle of deflection of the photon for $b = 3.2$, $b = 2.8$ and $b = 2.65$. What happens when $b = 2.5$?

References

[1] B. F. Schutz, *A first course in General Relativity*, CUP (1990).

