UNIVERSITY OF CAMBRIDGE
Faculty of Mathematics

MATHEMATICAL TRIPOS

2024-25

GUIDE TO COURSES IN PART IA

This booklet provides an introduction for new students, giving an outline of the first year with informal and non-technical descriptions of the courses.

This *Guide to Courses* is intended to supplement the more formal descriptions contained in the booklet *Schedules of Lecture Courses and Form of Examinations*.

These and other Faculty documents for students taking the Mathematical Tripos are available from the undergraduate pages on the Faculty's website at `https://www.maths.cam.ac.uk/undergrad/`

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1 Introduction

The *Mathematical Tripos* consists of Parts IA, IB and II, normally taken in consecutive years, with an optional fourth year, Part III, which can taken by students who do sufficiently well. Those who successfully complete three years are eligible to graduate with a BA honours degree, while those who go on to complete the additional fourth year graduate with both BA honours and MMath degrees.

The Mathematical Tripos is tightly structured, with no choice in the first year, some choice in the second year, and a very wide choice in the third year.

This booklet provides an introduction for new students, with an outline of Part IA and informal descriptions of the courses. The descriptions are intended to be comprehensible without much prior knowledge and to convey something of the flavour of each course (corresponding booklets are also available for Part IB and Part II).

You will also find it helpful to consult the booklet *Schedules of Lecture Courses and Form of Examinations*, available for download at `https://www.maths.cam.ac.uk/undergrad/course/schedules.pdf`, which is usually known as *the Schedules*. It describes the structure of the Tripos formally, in more technical detail, and it is the definitive reference for matters of course content and assessment, for students, lecturers and examiners. The introductory sections, in particular, should be read carefully alongside this booklet.

Your college Director of Studies will be able to provide further advice and guidance on all aspects of the Mathematical Tripos.

Lectures and examinations post COVID-19

On 5 May 2023 the WHO Director-General declared, with great hope, the end to COVID-19 as a global health emergency. Therefore, it is the working assumption of the Faculty that lectures and examinations in 2024-25 will all be held 'normally' and in person. Students are expected to attend lectures in order to take full advantage of the benefits of in-person teaching.

While arrangements for supervisions are made by Colleges and Directors of Studies, the Faculty anticipates that supervisions will also be held 'normally' and in person.

Changes since last year

There are no changes to the content of lecture courses in Part IA in 2024-25 compared to 2023-24.

2 The First Year in Outline

Students are admitted to study one of two options in Part IA: (*a*) Pure and Applied Mathematics; or (*b*) Mathematics with Physics. (You will have chosen one of these options when you applied.) There is no choice within each option: you are expected to follow all the courses.

- For option (*a*) there are eight 24-lecture courses (four in Michaelmas Term and four in Lent Term) and four 3-hour examination papers, with two courses examined on each paper.
- For option (*b*), the lecture courses Numbers & Sets and Dynamics & Relativity are replaced by the complete Physics course from Part IA of the Natural Sciences Tripos, which has lectures in Michaelmas, Lent and Easter Terms and assessed practical work throughout the year. Paper 4 of the Mathematics examination is replaced by the Physics paper from Natural Sciences.
- Students taking options (*a*) and (*b*) are classed together, as a single group, following the examinations at the end of the year.
- Options (*a*) and (*b*) do not extend into the second year. Those taking option (*a*) will (usually) continue to Part IB Mathematics; those taking option (*b*) may do the same, or they may change to Part IB Natural Sciences.

Option (b) is designed for students who have a strong interest in mathematics but who may wish to change to Part IB of the Natural Sciences Tripos (and take the Physics A / Physics B / Mathematics options) after the first year. It provides an excellent mathematical background for students who plan to study theoretical or experimental physics: their greater mathematical knowledge, compared with students who come to Physics through Part IA of the Natural Sciences Tripos, can be a significant benefit.

Changing from option (b) to option (a) is usually feasible if it is done early enough, but the courses cover ground rapidly; if this is something you are considering then you should discuss it with your Director of Studies as soon as possible. Changing from option (a) to option (b) is likely to be more complicated, because of the additional assessed practical work.

Michaelmas Term: non-examinable mechanics course

In Michaelmas Term there is a short non-examinable course on Mechanics. This is intended to provide catch-up material for those students who have taken only a limited amount of Mechanics at A-level (or the equivalent). You should discuss with your Director of Studies whether it might be sensible for you to attend these lectures.

Easter Term courses

A number of lecture courses are given in Easter Term, none of them examinable in Part IA.

- Lectures on Computational Projects (CATAM). It is essential to attend these in your first year, to prepare for the project work that will be submitted and assessed during your second year.
- Two courses examinable in Part IB can be attended in either your first or second year (or both): Variational Principles, and Optimisation.

One advantage of attending Part IB Easter Term courses in your first year is that material prepared well in advance can sink in much better than material prepared just before the exams. An additional benefit is that the Easter Term courses can provide helpful background for other courses in Part IB, e.g. the content of Variational Principles will connect well with topics covered in a number of other applied courses. It is common for Directors of Studies to advise their students to attend lectures in the Easter Term of their first year, work on examples sheets over the summer, and then have supervisions at the very start of their second year.

For the Computational Projects course, the Part IB CATAM manual is usually available at the end of July or the beginning of August at the end of your Part IA year. You should bear in mind that the computational work can be time-consuming and so an early start is strongly recommended. The two core projects require little mathematical knowledge from Part IB courses, so you can make substantial progress over the summer before the start of your second year.

Additional activities and lectures

Most students will find that there is enough mathematics in Part IA to keep them busy (or very busy!), and the Faculty places no expectations on students beyond keeping up with the first-year lectures, examples sheets and supervisions. There are many other educational and recreational opportunities to enjoy at university, though mathematics itself can hopefully be recreational.

For those who do want something extra or something a bit different from the mathematics in Part IA, one of the first choices could be the many excellent talks and lectures provided by the student maths societies. These provide an unparalleled opportunity to hear leading experts talking in an accessible way about some of the most important new ideas in mathematics and the mathematical sciences.

It is even possible to attend/preview some lectures in Part IB in your first year, but this must not be allowed to detract from the time and effort you devote to the IA courses and example sheets. Working to achieve a thorough understanding of the IA material, and developing more mature and subtle ways of thinking about mathematics in the process, is the key to making a successful transition from school to university.

The Faculty attempts to timetable the Part IB course Groups, Rings and Modules in the Lent Term so that it can be attended by first-year students. Hence, if you felt very comfortable with the workload in the Michaelmas Term, you might consider previewing this course (though it is generally unwise to have supervisions too - these are better left to next year when you will gain more benefit). As always, your Director of Studies will be able to provide guidance, e.g. they may advise you to concentrate on learning the Part IA courses as thoroughly as possible; that way, you would need less revision time in the Easter Term and this would allow you to take the courses provided then, i.e. Variational Principles and/or Optimisation.

3 Informal Descriptions of Courses in Part IA

Each lecture course has an official syllabus, or *schedule*, that sets out formally, and in technical terms, the material to be covered. The *schedules* are listed in the booklet *Schedules of Lecture Courses and Form of Examinations* that is available for download at `https://www.maths.cam.ac.uk/undergrad/course/schedules.pdf`.

This section, by contrast, provides an *informal* description of each lecture course given by the Faculty of Mathematics that is examinable in Part IA. Students taking option (b) Mathematics with Physics should refer to `https://www.phy.cam.ac.uk/students/teaching` for a description of the examinable material that replaces the courses Numbers & Sets and Dynamics & Relativity from option (a).

Each description below ends with a summary of the learning outcomes for the course. The full learning outcome for Part IA is that you should understand the material described in the formal syllabuses given in the *Schedules* booklet and be able to apply it to the sorts of problems that can be found on Tripos papers from earlier years.

Vectors and Matrices — 24 lectures, Michaelmas Term

Most students will be familiar with vectors and matrices to some extent; this course recaps the basics before introducing new ideas, results and techniques. The material is absolutely fundamental to nearly all areas of mathematics.

You will learn to deal with vectors in a rather general sense, allowing the description of points, lines and planes in 2 or 3 dimensions to be extended to higher, and even complex, dimensions. You will also enhance and refine your understanding of matrices: studying them as transformations or *linear maps*; using them to analyse incisively, and in full generality, sets of linear equations; and deriving new results and applications, e.g. the classification of quadric surfaces. A complete understanding of the material calls for a good balance between complementary points of view; in particular between algebra and geometry, but also between abstract ideas (e.g. what do we really mean by *dimension*?) and practical applications (e.g. how do we calculate determinants and matrix inverses?).

The course begins with a recap of complex numbers and their connection to geometry in the plane, via de Moivre's theorem, logarithms, and powers. We then move to three dimensions, discussing scalar and vector products and introducing index notation and the summation convention (due to Einstein) – this is used extensively in applied maths and theoretical physics. In parallel, key concepts applicable to all vectors such as *span*, *linear independence* and *basis* are introduced. As the course progresses, there is plenty of practice with matrix algebra, and we meet additional useful notions such as *rank*, *image* and *kernel*. Much of the last third of the course deals with the important concepts of *eigenvectors* and *eigenvalues* and discusses certain standard or *canonical forms* for matrices (how to describe a given transformation as simply as possible). Finally, there are some examples of groups of matrices, including the symmetry group of Special Relativity in two (*spacetime*) dimensions.

Learning outcomes. By the end of this course, you should:

- be able to manipulate complex numbers and be able to solve geometrical problems using complex numbers;

- be able to manipulate vectors in $\mathbb{R}^3$ (using index notation and summation convention where appropriate), and to solve geometrical problems using vectors;
- be able to manipulate matrices and determinants, and understand their relation to linear maps and systems of linear equations;
- be able to calculate eigenvectors and eigenvalues and understand their relation with diagonalisation of matrices and canonical forms.

Analysis I — 24 lectures, Lent Term

Analysis involves the rigorous investigation of limits and calculus. You need to study analysis to have a firm foundation for techniques you already know, such as basic differentiation and integration. This not only allows you to understand exactly when these techniques can be used, but also allows you to generalise them to more complicated situations.

The sorts of questions that you will be asking in this course are: 'what does it mean to say that a sequence or a function tends to a limit?'; 'what is the exact definition of a derivative or an integral?'; 'which functions can be differentiated and which can be integrated?'; 'what conditions are needed for a Taylor series to be valid?'.

In later courses on analysis, differentiation and integration of functions of more than one variable are investigated.

In Analysis I, you will encounter the 'ϵ-δ' method of characterising the properties of functions. This is the basis of rigorous thought in this area of mathematics, and will repay you handsomely for all the work you put into understanding it.

Learning outcomes. By the end of this course, you should:

- be able to apply the basic techniques of rigorous analysis and be familiar with examples of 'good behaviour' and 'bad behaviour' in basic analysis;
- know the definition of a limit and be able to establish the convergence or divergence of simple real and complex sequences and series;
- understand the completeness of the real line and be able to derive the basic properties of continuous real-valued functions;
- be able to establish the rules for differentiation, and to prove and apply the mean value theorem;
- be acquainted with complex power series and be able to determine the radius of convergence in simple cases;
- know the definition of the Riemann integral, be able to test simple functions for integrability, and establish the rules for integration.

Differential Equations — 24 lectures, Michaelmas Term

The main aim is to develop the skill of representing real (physical or biological) situations by means of differential (or difference) equations. The course follows smoothly from the A-level syllabus, starting with revision of differentiation and integration.

A particularly important sort of differential equation is one which is linear and has constant coefficients. These equations are unusual in that they can be solved exactly (the solutions are exponential or trigonometric functions). Many of the equations of physics are of this sort: the equations governing radioactive decay, Maxwell's equations for electromagnetism and the Schrödinger equation in quantum mechanics, for example.

In other cases, it is useful to try to represent solutions which cannot be obtained explicitly by means of phase-plane diagrams. Sometimes a particular solution describing some important situation is known although the general solution is not. In this case, it is often important to determine whether this solution is typical, or whether a small change in the conditions will lead to a very different solution. In the latter case, the solution is said to be unstable. This property is determined by linearising the original equation to obtain an equation with constant coefficients of the sort discussed above. Sometimes, the solutions are so unstable that they are called *chaotic*.

The very important idea of partial differentiation is also introduced in the course. This is the analogue of familiar differentiation to functions which depend on more than one variable. The approach is mainly geometrical and one of the applications is determining the stationary points of, for example, a function that gives height above sea-level and classifying them into maxima (mountain peaks), minima (valley bottoms) and saddle points (cols or passes).

Learning outcomes. By the end of this course, you should:

- understand the theory of, and be able to solve (in simple cases), linear differential or difference equations, and standard types of non-linear equations;
- calculate partial derivatives and use the chain rule;
- find and classify stationary points of functions of more than one variable;
- be able to investigate the stability of solutions of differential or difference equations.

Probability — 24 lectures, Lent Term

From its origin in games of chance and the analysis of experimental data, probability theory has developed into an area of mathematics with many varied applications in physics, biology and business.

This course introduces the basic ideas of probability and should be accessible to students who have no previous experience of probability or statistics. While developing the underlying theory, the course should strengthen students' general mathematical background and manipulative skills by its use of the axiomatic approach. There are links with other courses, in particular Vectors and Matrices, the elementary combinatorics of Numbers and Sets, the difference equations of Differential Equations and calculus of Vector Calculus and Analysis. Students should be left with a sense of the power of mathematics in relation to a variety of application areas.

After a discussion of basic concepts (including conditional probability, Bayes' formula, the binomial and Poisson distributions, and expectation), the course studies random walks, branching processes, geometric probability, simulation, sampling and the central limit theorem. Random walks can be used, for example, to represent the movement of a molecule of gas or the fluctuations of a share price; branching processes have applications in the modelling of chain reactions and epidemics. Through its treatment of discrete and continuous random variables, the course lays the foundation for the later study of statistical inference.

Learning outcomes. By the end of this course, you should:

- understand the basic concepts of probability theory, including independence, conditional probability, Bayes' formula, expectation, variance and generating functions;
- be familiar with the properties of commonly-used distribution functions for discrete and continuous random variables;
- understand and be able to apply the central limit theorem.
- be able to apply the above theory to 'real world' problems, including random walks and branching processes.

Groups 24 lectures, Michaelmas Term

In university mathematics, *algebra* is the study of abstract systems of objects whose behaviour is governed by fixed rules or *axioms*. An example is the set of real numbers, governed by the rules of addition and multiplication. One of the simplest forms of abstract algebraic systems is a group, which is roughly a set of objects and a rule for multiplying them together. Groups arise all over mathematics, particularly where there is symmetry.

This course introduces groups and their properties. The emphasis is on both the general theory and a range of examples, such as groups of symmetries and groups of linear transformations.

Learning outcomes. By the end of this course, you should:

- be familiar with elementary properties of abstract groups, including the theory of mappings between groups;
- understand the group-theoretic perspective on symmetries in geometry.

Vector Calculus 24 lectures, Lent Term

This course is about functions of more than one variable. It is an 'applied' course, meaning that you are expected to be able to apply techniques, but not necessarily to prove rigorously that they work – that will come in future analysis courses.

In the first part of the course, the idea of integration is extended from $\mathbb{R}$ to $\mathbb{R}^2$ and $\mathbb{R}^3$ (with an obvious extension to higher dimensions): integrals along the x-axis are replaced by integrals over curves, surfaces and volumes.

Then the idea of differentiation is extended to vectors (div, grad and curl), which is a basic tool in many areas of theoretical physics (such as electromagnetism and fluid dynamics).

Two important theorems are introduced, namely the divergence theorem and Stokes's theorem; in both cases, an integral over a region (in $\mathbb{R}^3$ and in $\mathbb{R}^2$, respectively) is converted to an integral over the boundary of the region.

All the previous ideas are then applied to Laplace's equation $\nabla^2\phi = 0$ and the related Poisson's equation, which are amongst the most important equations in all of mathematics and physics.

Finally, the notion of a vector is generalised to that of a *tensor*. A vector can be thought of as a 3×1 matrix that carries physical information: namely, magnitude and direction. This information is preserved when the axes are rotated only if the components change according to a certain rule. Very often, it is necessary to describe physical quantities using a 3×3 matrix (or even a $3 \times 3 \times 3 \cdots$ 'matrix'). Such a quantity is called a tensor if its components transform according to a certain rule when the axes are rotated. This rule means that the physical information embodied in the tensor is preserved.

Learning outcomes. By the end of this course, you should:

- be able to manipulate, and solve problems using, vector operators;
- be able to calculate line, surface and volume integrals in $\mathbb{R}^3$, using Stokes theorem and the divergence theorem where appropriate;
- be able to solve Laplace's equation in simple cases, and be able to prove standard uniqueness theorems for Laplace's and related equations.
- understand the notion of a tensor and the general properties of tensors in simple cases.

Numbers and Sets — 24 lectures, Michaelmas Term

This course is concerned not so much with teaching you new parts of mathematics as with explaining how the language of mathematical arguments is used. We will use simple mathematics to develop an understanding of how results are established.

Because you will be exploring a broader and more intricate range of mathematical ideas at university, you will need to develop greater skills in understanding arguments and in formulating your own. These arguments are usually constructed in a careful, logical way as proofs of propositions. We begin with clearly stated and plausible assumptions or *axioms* and then develop a more and more complex theory from them. The course, and the lecturer, will have succeeded if you finish the course able to construct valid arguments of your own and to examine critically those that are presented to you. Example sheets and supervisions will play a key role in achieving this. These skills will form the basis for the later courses, particularly those devoted to Pure Mathematics.

In order to give examples of arguments, we will take two topics: sets and numbers. Set theory provides a basic vocabulary for much of mathematics. We can use it to express in a convenient and precise shorthand the relationships between different objects. Numbers have always been a fascinating and fundamental part of Mathematics. We will use them to provide examples of proofs, algorithms and counter-examples.

Initially we will study the natural numbers $1, 2, 3, \ldots$ and especially *mathematical induction*. Then we expand to consider integers and arithmetic leading to codes like the RSA code used on the internet. Finally we move to rational, real and complex numbers where we lay the logical foundations for analysis. (Analysis is the name given to the study of, for example, the precise meaning of differentiation and integration and the sorts of functions to which these processes can be applied.)

Learning outcomes. By the end of this course, you should:

- understand the need for rigorous proof in mathematics, and be able to apply various different methods, including proof by induction and contradiction, to propositions in set theory and the theory of numbers;
- know the basic properties of the natural numbers, rational numbers and real numbers;
- understand elementary counting arguments and the properties of the binomial coefficients;
- be familiar with elementary number theory and be able to apply your knowledge to the solution of simple problems in modular arithmetic;
- understand the concept of countability and be able to identify typical countable and uncountable sets.

Dynamics and Relativity — 24 lectures, Lent Term

This course assumes knowledge from A-level mechanics (or the equivalent). If you are unsure whether you have the necessary background, then you should attend at least the first lecture of the non-examinable introductory Mechanics course in the Michaelmas Term.

This course is the first look at theoretical physics. The course is important not just for the material is contains; it is also important because it serves as a model for the mathematical treatment of all later courses in theoretical physics.

The first 17 or so lectures are on classical dynamics. The basis of the treatment is the set of laws due to Newton that govern the motion of a particle under the action of forces, and which can be extended to solid bodies. The approach relies heavily on vector methods.

One of the major topics is motion in a gravitational field. This is not only an important application of techniques from this course and the Differential Equations course, it is also of historical interest: it was in order to understand the motions of the planets that Isaac Newton developed calculus.

With the advent of Maxwell's equations in the late nineteenth century came a comfortable feeling that all was well in the world of theoretical physics. This complacency was rudely shaken by Michelson's attempt to measure the velocity of the Earth through the surrounding aether by comparing the speed of light measured in perpendicular directions. The surprising result was that it makes no difference whether one is travelling towards or away from the light source; the velocity of light is always the same. Various physicists suggested a rule of thumb (time dilation and length contraction) which would account for this phenomenon, but it was Einstein who deduced the underlying theory, special relativity, from his considerations of the Maxwell equations.

In this short introduction, the last 7 or so lectures of this course, there is time only to develop the framework in which the theory can be discussed (the amalgamation of space and time into Minkowski space-time) and tackle simple problems involving the kinematics and dynamics of particles.

Learning outcomes. By the end of this course, you should:

- appreciate the axiomatic nature of, and understand the basic concepts of, Newtonian mechanics;
- be able to apply the theory of Newtonian mechanics to simple problems including the motion of particles, systems of particles and rigid bodies, collisions of particles and rotating frames;
- be able to calculate orbits under a central force and investigate their stability;
- be able to tackle problems in rotating frames;
- be able to solve relativistic problems involving space-time kinematics and simple dynamics.

Mechanics (a non-examinable introduction) 10 lectures, Michaelmas Term

This course covers the background material in mechanics required for the Dynamics and Relativity course in the Lent Term (and for later courses in applied mathematics and theoretical physics). It is intended for students who have taken only a limited amount of mechanics at A-level (or the equivalent), e.g. students who have not taken mechanics as part of Further Maths A-level. You should attend at least the first lecture if you are unsure whether you have covered the right material. Each of the lectures will discuss an important topic, such as conservation of momentum or conservation of energy, including worked examples, and each topic will be announced beforehand, so you can decide whether you should attend. The course should not require a significant investment of time.

4 Computational Projects (CATAM)

The lectures for this course should be attended in the Easter Term of the first year.

The Computational Projects course (CATAM) consists mainly of practical projects, with an emphasis on understanding the physical and mathematical problems being modelled rather than on the details of computer programming. Projects must be written up and submitted during the second year (with deadlines just after the start of Lent and Easter Terms) and marks contribute to the total result for the Part IB examination. Lectures are given in the Easter Term of the first year to introduce some of the mathematical and practical aspects of the various projects. This allows an early start to be made on CATAM over the summer, which is strongly recommended. More details are available in the *Part IB Computational Projects Manual*, which is online at `https://www.maths.cam.ac.uk/undergrad/catam/IB`. The supported programming language is MATLAB.

Learning outcomes. By the end of this course, you should:

- be able to programme using a traditional programming language;
- understand the limitations of computers in relation to solving mathematical problems;
- be able to use a computer to solve problems in both pure and applied mathematics involving, for example, solution of ordinary differential equations and manipulation of matrices.

5 Informal Descriptions of Part IB Easter Term Courses

The following courses are lectured in the Easter Term and examined in Part IB.

Optimisation — 12 lectures, Easter Term

This course may be taken in the Easter Term of either the first or the second year.

A typical problem in optimisation is to find the cheapest way of supplying a set of supermarkets from a set of warehouses: in more general terms, the problem is to find the minimum (or maximum) value of a quantity when the variables are subject to certain constraints. Many real-world problems are of this type and the theory discussed in the course are practically extremely important as well as being interesting applications of ideas introduced earlier in Numbers and Sets and Vectors and Matrices.

The theory of Lagrange multipliers, linear programming and network analysis is developed. Topics covered include the simplex algorithm, the theory of two-person games and some algorithms particularly well suited to solving the problem of minimising the cost of flow through a network.

Learning outcomes. By the end of this course, you should:

- understand the nature and importance of convex optimisation;
- be able to apply Lagrangian methods to solve problems involving constraints;
- be able to solve problems in linear programming by methods including the simplex algorithm and duality;
- be able to solve network problems by methods using, for example, the Ford–Fulkerson algorithm and min-cut max-flow theorems.

Variational Principles — 12 lectures, Easter Term

This course may be taken in the Easter Term of either the first year or the second year; however it contains helpful background material for many of the other applied courses in Part IB.

The techniques developed in this course are of fundamental importance throughout physics and applied mathematics, as well as in many areas of pure and applicable mathematics.

The first part of the course considers stationary points of functions on $\mathbb{R}^n$ and extends the treatment in Part IA Differential Equations to deal with *constraints* using the method of *Lagrange multipliers*. This allows one to determine e.g. the stationary points of a function on a surface in $\mathbb{R}^3$.

The second part of the course deals with *functionals* (and functional derivatives) and enables one to find the path that minimises the distance between two points on a given surface (a *geodesic*), the path of a light ray that gives the shortest travel time (satisfying *Fermat's Principle*), or the minimum energy shape of a soap film.

Many fundamental laws of physics (in Newtonian mechanics, relativity, electromagnetism or quantum mechanics) can be expressed as variational principles in a profoundly elegant and useful way that brings underlying symmetries to the fore.

Learning outcomes. By the end of this course, you should:

- understand the concepts of a functional, and of a functional derivative;
- be able to apply constraints to variational problems;
- appreciate the relationship between variational statements, conservation laws and symmetries in physics.