Basics.

1. In the usual formalization of natural numbers and ordered pairs (i.e., \(n = \{0, \ldots, n-1\}\) and \((x, y) := \{\{x\}, \{x, y\}\}\)), one of the following statements is true. Which one?
   \[\square \text{ A } 17 \in 4. \]
   \[\square \text{ B } (0, 1) = 2. \]
   \[\square \text{ C } 2 \in (0, 1). \]
   \[\square \text{ D } 4 \in (4, 17). \]

2. We often use informal mathematical notation using curly braces to denote sets. However, not every expression corresponds to a set; sometimes, we denote proper classes. Among the following expressions, one corresponds to a proper class. Which one?
   \[\square \text{ A } \{x \mid x \text{ is a nonempty subset of the natural numbers}\}. \]
   \[\square \text{ B } \{x \mid x \text{ is a finite set of real numbers}\}. \]
   \[\square \text{ C } \{x \mid x \text{ is a one-element set of rational numbers}\}. \]
   \[\square \text{ D } \{x \mid x \text{ is a two-element subset of a vector space}\}. \]

3. Consider the following model \(M = (\{x, y\}, \in)\) as a model of set theory (where \(x \in x\) and \(x \in y\), but not \(y \in x\) or \(y \in y\)): 
   \[\xymatrix{ x & y \ar@{~}[l] }\]
   One of the following axiom (scheme)s of set theory is true in \(M\). Which one?
   \[\square \text{ A } \text{ The Pairing Axiom.} \]
   \[\square \text{ B } \text{ The Axiom of Foundation.} \]
   \[\square \text{ C } \text{ The Union Axiom.} \]
   \[\square \text{ D } \text{ The Axiom Scheme of Separation.} \]

4. The Zermelo numbers are defined by the following recursion: \(z_0 := \emptyset\) and \(z_{n+1} := \{z_n\}\). The von Neumann numbers are defined by: \(v_0 := \emptyset\) and \(v_{n+1} := v_n \cup \{v_n\}\) (i.e., \(v_n = n\)). One of the following statements is true. Which one?
   \[\square \text{ A } z_2 = v_2. \]
   \[\square \text{ B } z_2 \in v_4. \]
   \[\square \text{ C } z_2 \subseteq v_2. \]
   \[\square \text{ D } z_2 \in z_2. \]
5. One of the following statements about \((X, R)\) implies that \((X, R)\) is a wellorder. Which one?

- □ A  \(R\) is a transitive relation.
- □ B  \(R\) is a linear relation.
- □ C  \(X\) has an \(R\)-minimal element.
- □ D  There is a wellorder \((Y, S)\) and an injective function \(f : X \to Y\) such that for all \(x_0, x_1 \in X\), we have \(x_0 R x_1\) if and only if \(f(x_0) S f(x_1)\).

6. Consider the integers \(\mathbb{Z}\) with their natural order \(<\) and their natural multiplication \(\cdot\). One of the following sets is wellordered by \(<\). Which one?

- □ A  \(\mathbb{Z}\setminus \{0\}\),
- □ B  \(\{z \in \mathbb{Z}; \exists x \in \mathbb{Z}(z = 2 \cdot x)\}\),
- □ C  \(\{z \in \mathbb{Z}; z < 0 \land \exists x \in \mathbb{Z}(z = 2 \cdot x)\}\),
- □ D  \(\{z \in \mathbb{Z}; \exists x \in \mathbb{Z}(z = x \cdot x)\}\).

**Ordinals.**

7. One of the following is provable in ZF. Which one?

- □ A  Every transitive set is an ordinal.
- □ B  Every transitive set of ordinals is an ordinal.
- □ C  Every set of ordinals is transitive.
- □ D  None of the above.

8. Only one of the following statements is correct. Which one?

- □ A  There are two different order isomorphisms between \(\omega_1\) and \(\omega_1\).
- □ B  There are two different order isomorphisms between \(\omega\) and \(\omega\).
- □ C  There are two different order-preserving embeddings from \(\omega\) to \(\omega_1\).
- □ D  There are two different order-preserving embeddings from \(\omega_1\) to \(\omega\).

9. One of the following ordinal inequalities is true. Which one?

- □ A  \(5 \cdot \omega < \omega \cdot 5\).
- □ B  \(\omega \cdot 5 < 5 \cdot \omega + 5\).
- □ C  \(5 + \omega + \omega \cdot \omega < \omega + 5 + \omega \cdot \omega \cdot \omega\).
- □ D  \(5 \cdot (20 + \omega_1) < 5 \cdot \omega_1\).
10. The statement “there are no cardinal numbers between $\aleph_0$ and $\aleph_1$” is...
   - □ A ...provable in ZF,
   - □ B ...provable in ZFC, but not in ZF,
   - □ C ...equivalent to the Continuum Hypothesis in the base theory ZFC.
   - □ D None of the above.

11. In ZFC, one of the following statements is equivalent to the continuum hypothesis. Which one?
   - □ A There is a bijection between the power set of $\mathbb{N}$ and the real numbers.
   - □ B Every set of real numbers has an uncountable subset.
   - □ C Every uncountable set of real numbers has a countable subset.
   - □ D Every uncountable set of real numbers has a subset that is in bijection with the real numbers.

12. Work in ZF and consider $W := \{(A, R) ; A \subseteq \mathbb{N} \text{ and } (A, R) \text{ is a wellorder}\}$. What is the cardinality of the set $W$?
   - □ A ZF proves that it is $\aleph_0$.
   - □ B ZF proves that it is $\aleph_1$.
   - □ C ZF does not prove that it is $\aleph_1$, but ZFC proves that it is $\aleph_1$.
   - □ D ZFC does not prove that it is $\aleph_1$, but ZFC + CH proves that it is $\aleph_1$.

13. Only one of the following statements is false. Which one?
   - □ A There is an order preserving injection from $\omega$ to $\aleph_\omega$.
   - □ B There is an order preserving injection from $\omega_1$ to $\aleph_\omega$.
   - □ C There is a cofinal order preserving injection from $\omega$ to $\aleph_\omega$.
   - □ D There is a cofinal order preserving injection from $\omega_1$ to $\aleph_\omega$.

14. Let $\kappa$ be a cardinal with $\text{cf}(\kappa) = \aleph_0$. Only one of the following statements is provable. Which one?
   - □ A Every injective function from $\kappa$ to $\kappa$ has countable range.
   - □ B The set $\kappa$ is a countable union of sets of cardinality strictly smaller than $\kappa$.
   - □ C The cardinal $\kappa$ is countable.
   - □ D If $\kappa = \aleph_\xi$, then $\xi$ is a countable ordinal.