The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This graduate course will provide an introduction to the basic ideas of the subject.

Suppose $\Gamma$ is a discrete group of isometries of a metric space $X$. We focus on the theorems we can prove about $\Gamma$ by imposing geometric conditions on $X$. These conditions are motivated by curvature conditions in differential geometry, but apply to general metric spaces and are much easier to state. First we study the case when $X$ is Gromov-hyperbolic, which corresponds to negative curvature. Then we study the case when $X$ is $\text{CAT}(0)$, which corresponds to non-positive curvature. In order for this theory to be useful, we need a rich supply of negatively and non-positively curved spaces. We develop the theory of non-positively curved cube complexes, which provide many examples of CAT(0) spaces and have been the source of some dramatic developments in low-dimensional topology over the last twenty years.

Part 1. We will introduce the basic notions of geometric group theory: Cayley graphs, quasi-isometries, the Schwarz–Milnor Lemma, and the connection with algebraic topology via presentation complexes. We will discuss the word problem, which is quantified using the Dehn functions of a group.

Part 2. We will cover the basic theory of word-hyperbolic groups, including the Morse lemma, local characterization of quasigeodesics, linear isoperimetric inequality, finitely presentedness, quasiconvex subgroups etc.

Part 3. We will cover the basic theory of CAT(0) spaces, working up to the Cartan–Hadamard theorem and Gromov’s Link Condition. These two results together enable us to check whether the universal cover of a complex admits a CAT(0) metric.

Part 4. We will introduce cube complexes, in which Gromov’s link condition becomes purely combinatorial. If there is time, we will discuss Haglund–Wise’s special cube complexes, which combine the good geometric properties of CAT(0) spaces with some strong algebraic and topological properties.

Pre-requisites

Part IB Geometry and Part II Algebraic topology are required.

Literature


**Additional support**

As a graduate course, none of this material is examinable. Nevertheless, the four example sheets from a previous Part III version of this course may provide a useful resource.