UNIVERSITY OF CAMBRIDGE
Faculty of Mathematics

Mathematical Tripos
Part III Guide to Courses 2016-2017

The Faulkes Institute of Geometry, completed in January 2002

Mathematical Tripos
Part III Lecture Courses in 2016-2017

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2016-17. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the Guide to Courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/courseguide.html) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.

Table of Contents

Algebra

Lie Algebras and their representations (M24)

Ian Grojnowski

This course is an introduction to the basic properties of finite dimensional complex Lie algebras and of their representations.

Lie algebras are 'infinitesimal symmetries'; linearisations of groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry, and in theoretical physics (string theory, exactly solvable models in statistical mechanics),...

One reason for their importance is that the finite dimensional complex representations of the simple Lie algebras are exactly the same as those of the corresponding groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can use nothing other than linear algebra and still completely describe these representations.

(Later you will want to study the topology and geometry as well, of course!)

We will cover the following topics:

Definitions, motivations, and basic structure theory.

Root systems, Weyl groups, the finite simple Lie algebras.

Classification of finite dimensional representations, Verma modules, Weyl character formula.

Crystals, Littelmann paths.

If there is time, we will finish by discussing affine Lie algebras, the basic representation, Boson-Fermion correspondence, and theta functions.

Desirable Previous Knowledge

None other than linear algebra, but the part II course on representation theory (or equivalent) will be useful as background.

Reading to complement course material

1. V. Kac, Infinite dimensional Lie algebras, Cambridge University Press
2. M. Kashiwara. On crystal bases. *in* Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16,
3. N. Jacobson. Lie algebras

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Algebras (L24)

C.J.B. Brookes

The aim of the course is to give an introduction to algebras. The emphasis will be on non-commutative examples that arise in representation theory (of groups and Lie algebras) and the theory of algebraic D-modules, though you will learn something about commutative algebras in passing.

Topics I hope to fit in are:

Artinian algebras. Examples, group algebras of finite groups, crossed products. Structure theory. Artin-Wedderburn theorem. Projective modules. Blocks.

Noetherian algebras. Examples, quantum plane and quantum torus, differential operator algebras, enveloping algebras of finite dimensional Lie algebras. Structure theory. Injective hulls, uniform dimension and Goldie's theorem.

Hochschild chain and cochain complexes. Hochschild homology and cohomology. Gerstenhaber algebras.

K_0 and K_1.

Deformation of algebras. Quantum co-ordinate algebras and quantum enveloping algebras.

Coalgebras, bialgebras and Hopf algebras. Quantum groups.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields, or III Lie algebras will be helpful but not necessary.

Literature

1. DJ Benson, *Representations and cohomology.* volumes 1 and 2, Cambridge studies in advanced mathematics 30 and 31, C.U.P.

2. KR Goodearl and RB Warfield, *An introduction to non-commutative Noetherian rings.* LMS student texts, C.U.P.

3. JC McConnell JC and JC Robson, *Noncommutative Noetherian rings.* 1st edition Wiley, 2nd edition A.M.S. graduate studies series.

4. RS Pierce, *Associative Algebras.* Graduate Texts in Mathematics 88, Springer.

Additional support

Four examples sheets will be provided, with supporting examples classes.

Representation Theory(L24)

S. Martin

The representation theory of the symmetric group S_n is a classical subject that, from the foundational work of Frobenius, Schur and Young, has developed into a richly diverse area, with important connections across algebra, computer science, statistical mechanics and theoretical physics.

This course is essentially an introduction to the algebraic combinatorics that underpins a modern treatment of the representation theory of S_n. I hope to cover a selection of classical topics from amongst: Specht modules, Young symmetrizers, Young tableaux, (Young-)Jucys-Murphy elements (as generators of the GZ-algebra), the branching rule, Schur functions, the Robinson-Schensted-Knuth correspondence, the Jacobi-Trudi identity, the hook-lengths formula, the Littlewood -Richardson rule, the Murnaghan-Nakayama rule, Schützenberger's involution and jeu de taquin, etc. My approach to computing the complex finite-dimensional irreducible representations of S_n was developed by Anatoly Vershik and Andrei Okounkov (see the three-authored book [1] below) whose main tool is the so-called Gelfand-Tsetlin (GZ) algebra, which is a certain commutative subalgebra of $\mathbf{C}S_n$.

If time allows, some of the following more recent topics will be included: Hecke algebras, partition algebras, Macdonald polynomials, non-commutative symmetric functions. An excellent modern treatment of the combinatorial methodology is Stanley's book [6] in the preliminary reading and the tome [4] in the literature.

- Representations of S_n. The Young diagram, Young tableaux and standardness; the Young poset and the Bratelli diagram.
- Coxeter transpositions, wiring diagrams and Coxeter relations.
- The Okounkov-Vershik construction. Branching graph and Gelfand-Tsetlin bases. Centre of the group algebra. Young-Jucys-Murphy elements. YJM elements saitisfy generic Hecke algebra relations. Branching is multiplicity-free.

  I will then choose from the following selection of applications:
- Hook-lengths formula and hook-walks.
- Induced representations and the Frobenius-Young correspondence; semistandard Young tableaux and Kostka numbers
- Robinson-Schensted-Knuth (RSK) correspondence. Viennot's shadow construction for RSK.
- (if time) The Littlewood-Richardson rule and its variants.
- (if time) The ordinary characters of the symmetric group.

Prerequisites

Prerequisites are minimal. Undergraduate representation theory (semisimplicity of the complex group algebra, completeness of characters over **C**), permutation representations. Group theory (symmetric groups and general linear groups and their conjugacy classes).

Preliminary Reading

1. T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Representation theory of the symmetric groups: the Okounkov-Vershik approach, character formulas and partition algebras, CUP 2010.
2. W. Fulton, Young tableaux, Cambridge University Press, 1997.
3. W. Fulton and J. Harris, Representation theory, a first course, GTM 129, Springer, 1991.
4. G.D. James, The Representation Theory of the Symmetric Group, LNM 682, Springer 1978.
5. B.E. Sagan, The Symmetric Group: representations, combinatorial algorithms and symmetric functions (2nd edn), GTM 203, Springer 2001.
6. R.P. Stanley, Algebraic Cominatorics, UTM (Springer) 2013.

Literature

1. A. Garsia, Young's seminormal representation, Murphy elements and content evaluations, 2003. (See Garsia's UCSD webpage)
2. A. Garsia, Alfred Young's construction of the irreducible representations of S_n, 2014. Available online.
3. A. Lascoux. Young's representations of the symmetric group. Available online at http:// phalanstere.univ-mlv.fr/∼ al/ARTICLES/ProcCrac.ps.gz
4. R.P. Stanley, Enumerative Combinatorics, Volume 2 (Chapter 7), CUP 2001.

Additional support

Two sheets of examples will be provided backed up by two or three classes.

Decision Problems in Group Theory (L24)

Non-Examinable (Graduate Level)

Maurice Chiodo

Overview

The aim of this course is to investigate incomputable problems in group theory. We will define several different methods of computation and use these to show various problems to be incomputable, as well as determine the degrees of incomputability of such problems.

Course description

Computability theory: Turing machines, recursive and recursively enumerable sets, the halting set $\mathbb{K}$. Church's thesis. Many-one and Turing reductions, Kleene hierarchy. Modular machines and their equivalence to Turing machines. Minsky machines and how they simulate Turing machines. First order theories.
Group theory: Recursive presentations of groups, the word and isomorphism problems with basic properties and examples. Basic computable properties of groups. Post's construction of a finitely presented semigroup with unsolvable word problem.
Embedding theorems: A finitely presented group with unsolvable word problem; Higman's embedding theorem. A universal finitely presented group. The Adian-Rabin construction, unrecognisability of Markov properties. The Boone–Higman theorem. Properties preserved by Higman embeddings. Degrees of various incomputable problems.
Finite quotients: Slobodskoĭ's theorem on undecidability of the first order theory of finite groups. Bridson–Wilton theorem on undecidability of finite quotients.

Pre-requisites

It will be assumed that you have attended a first course in group theory, and that you attend at least the first 4 lectures of Part III Geometric Group Theory (Lent). In addition, Part II Automata and Formal Languages (Michaelmas) and/or Part III Logic (Lent) would be helpful for some intuition in computability theory, but are not essential.

Literature

1. R.I. Soare, *Recursively enumerable sets and degrees: a study of computable functions and computably generated sets.* Springer-Verlag (Perspectives in mathematical logic), 1987.
2. C. F. Miller III, *Decision Problems For Groups-Survey and Reflections.* Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., **23**, Springer, New York, 1–59 (1992). Also available at
   `http://www.ms.unimelb.edu.au/~cfm/papers/paperpdfs/msri_survey.all.pdf`
3. J. Rotman, *An Introduction To The Theory Of Groups.* (GTM 148), Springer, fourth edition, 1995.
4. D. E. Cohen, *Combinatorial Group Theory: A Topological Approach.* London Mathematical Society Student Texts, Cambridge University Press, 1989.
5. A. M. Slobodskoĭ, *Undecidability of the universal theory of finite groups* (Russian), Algebra i Logika **20**, no. 2, 207–230 (1981). English transl., Algebra and Logic **20**, no. 2, 139–156 (1981).
6. M. Bridson, H. Wilton, *The triviality problem for profinite completions*, Invent. Math. **202**, no. 2, 839–874 (2015).

Analysis

Analysis of Partial Differential Equations (M24)

Prof. Dafermos and Prof. Mouhot

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to modern functional analytic techniques, relying on a priori estimates, rather than explicit solutions, although the interaction with classical methods (such as the fundamental solution and Fourier representation) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. Some non-linear equations may also be discussed. The course will end with a discussion of major open problems in PDEs.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Preliminary Reading

The following article gives an overview of the field of PDEs:

1. Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Literature

1. Some lecture notes are available online at: `http://cmouhot.wordpress.com/teachings/`.

   The following textbooks are excellent references:

2. Evans, L. C., *Partial Differential Equations*, Springer, 2010.
3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.
4. John, F., *Partial Differential Equations*, Springer, 1991.

Additional Information

This course is also intended for doctoral students of the Centre for Analysis (CCA), who will also be involved in additional assignments, presentations and group work. Part III students do not do these, and they will be assessed in the usual way by exam at the end of the academic year. Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be one office hour a week.

Functional Analysis (M24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We will cover the following topics:

Hahn–Banach Theorem on the extension of linear functionals. Locally convex spaces.

Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.

Hahn–Banach Theorem on separation of convex sets. Extreme points and the Krein–Milman theorem. Partial converse and the Banach–Stone Theorem.

Banach algebras, elementary spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Some additional topics time permitting. For example, the Fréchet–Kolmogorov Theorem, weakly compact subsets of $L_1(\mu)$, the Eberlein–Šmulian and the Krein–Šmulian theorems, the Gelfand–Naimark–Segal construction.

Pre-requisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (much of which will be recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy's Theorem, Cauchy's Integral Formula and the Maximum Modulus Principle.

Literature

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales).* Oxford University Press, 2011.
2. Bollobás, Béla *Linear analysis: an introductory course.* Cambridge University Press, 1990.
3. Rudin, Walter *Real & Complex Analysis.* McGraw-Hill, 1987.
4. Rudin, Walter *Functional Analysis.* McGraw-Hill, 1990.
5. Taylor, S. J. *Introduction to measure and integration.* Cambridge University Press 1973.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be some material as well as examples sheets and announcements available at `www.dpmms.cam.ac.uk/~az10000/`

Topics in Ergodic Theory (M24)

Péter Varjú

Ergodic theory studies dynamical systems that are endowed with an invariant measure. There are many examples of such systems that originate from other branches of mathematics. This led to a fruitful interplay between ergodic theory and other fields, especially number theory.

I will explain some basic elements of ergodic theory, such as recurrence, ergodic theorems, mixing properties and entropy. I will also talk about some applications of the theory, such as Furstenberg's proof of Szemerédi's theorem, and Weyl's equidistribution theorem for polynomials.

I aim to cover the following topics:

- Furstenberg's correspondence principle,
- Poincaré recurrence, ergodicity,
- ergodic theorems,
- unique ergodicity,
- Weyl's equidistribution theorem for polynomials,
- mixing and weak mixing,
- the multiple recurrence theorem for weak mixing systems,
- entropy and its relation to mixing,
- Rudolph's theorem on $\times 2$, $\times 3$ invariant measures.

Pre-requisites

Measure theory, basic functional analysis, conditional expectation, Fourier transform.

Literature

Notes will be available on the lecturer's webpage.

Elliptic Partial Differential Equations (L24)

Neshan Wickramasekera

This course is intended as an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in geometric analysis. A strong background in the linear theory provides a foundation for understanding a number of non-linear problems including minimal submanifolds, harmonic maps, and general relativity. We will discuss both classical and weak solutions to linear elliptic equations focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. This involves establishing maximum principles, Schauder estimates, and other a priori estimates for the solutions. As time permits, we will discuss other topics including the De Giorgi-Nash theory (which provides the Harnack inequality and establishes Hölder continuity for weak solutions) and applications of the linear theory to quasilinear elliptic equations.

Pre-requisites

Lebesgue integration, Lebesgue spaces, Sobolev spaces, and basic functional analysis.

Literature

1. David Gilbarg and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).
2. Lawrence Evans, Partial Differential Equations. AMS (1998)
3. Qing Han and Fanghua Lin, Elliptic partial differential equations. Courant Lecture Notes, Vol. 1 (2011).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Convex Analysis (L24)

Non-Examinable (Graduate Level)

Dr Garling

This is a basic course on convex sets and convex functions. The results are of use in all areas of analysis.

1. The separation theorem and the Hahn-Banach theorem.
2. Subdifferentials, the Legendre transform and Orlicz spaces.
3. Differentiablity: Theorems of Mazur and Rademacher.
4. Ekeland's variational principle, and the Bishop-Phelps theorem .
5. Compact convex sets: the Krein-Mil'man theorem, dentability and fixed point theorems.
6. Introduction to Choquet theory.

Pre-requisites

Some knowledge of elementary functional analysis and probability theory is required.

Literature

1. Robert R. Phelps *Convex Functions, Monotone Operators and Differentiability.* Springer Lecture Notes in Mathematics 1364, 1993.
2. Robert R. Phelps *Lectures on Choquet Theory.* Springer Lecture Notes in Mathematics 1757, 2008.
3. R.Tyrrell Rockafellar *Convex Analysis.* Princeton, 1996.
4. Barry Simon *Convexity: an Analytic Viewpoint.* CUP, 2011.
5. Stephen Simons *From Hahn-Banach to Monotonicity.* Springer Lecture Notes in Mathematics 1693, 1998.

Variational Methods and PDE (E12)

Non-Examinable (Graduate Level)

Dr P. Markowich

The course deals with variational (minimization) problems in integral form and their associated Euler-Lagrange partial differential equations. Topics are lower semicontinuity, coericivity convexity, poly-convexity of functionals, obstacle problems, Hamiltonian-Lagrangian duality, gradient flows and the mountain pass theorem.

Pre-requisites

Partial differential equations, functional analysis and Sobolev spac

Literature

1. Lawrence C. Evans *Partial Differential Equations: Second Edition.*
2. *Graduate Studies in Mathematics Volume: 19* 2010; 749 pp

Advanced topics in many-particle systems (E12)

Non-Examinable (Part III Level)

Clément Mouhot

This non-examinable course will present some mathematical tools and concepts for the rigorous derivation and study of nonlinear partial differential equations arising from many-particle limits: Vlasov transport equations, Boltzmann collision equations, nonlinear diffusion, quantum Hartree equations... Depending on time and interest it will include part or all of the following items: Liouville and master equations of a many-particle system, the notion of empirical measures, the BBGKY hierarchy, the Hewitt-Savage theorem, the Braun-Hepp-Dobrushin theorem, the coupling method, the concepts of chaos and entropic chaos, the hydrodynamic limit of lattice systems.

Pre-requisites

Basics in measure theory, functional analysis, partial differential equations and probability.

Literature

1. H. Spohn, *Large Scale Dynamics of Interacting Particles.* Springer 1991.
2. C. Kipnis & C. Landim, *Scaling Limits of Interacting Particle Systems.* Springer 1999.
3. F. Golse, *The Mean-Field Limit for the Dynamics of Large Particle Systems*, Journeées Équations aux dérivées partielles Forges-les-Eaux, 2-6 juin 2003.
4. F. Bolley, *Optimal coupling for mean field limits*, available online
5. S. Mischler & C. Mouhot, *Kac's program in kinetic theory*, Inventiones mathematicae 2013, vol 193, pp 1-147.
6. S. Mischler, *Lecture notes on the propagation of chaos*, available online
7. P.-E. Jabin, *A review of the mean field limits for Vlasov equations*, Kinetic and Related Models 2014, vol 7, pp 661 - 711.

Function Spaces (L24)

Non-Examinable (Graduate Level)

Sophia Demoulini

Review (as necessary, including Measure Theory and Lebesgue spaces, Riesz representations on spaces of continuous functions, Egorov, Lusin BV spaces in 1 dim)

Hardy-Littlewood Principle Calderon-Zygmund decomposition Weak and Strong (p,q) operators Covering Theorems (Vitali, Besicovich) Absolute Continuity Differentiation Hahn and Lebesgue Decomposition

(Possibly: Introduction to Fractional Integral Operators) Hardy-Littlewood-Sobolev theorem)

Sobolev Spaces Hausdorff Measure - Isodiametric inequality Capacity Area and Coarea formulae Trace Capacity Space of functions of Bounded Variation in n-dimensions Coarea for BV Reduced boundary and Gauss-Green Theorem for BV

Pre-requisites

Measure theory, Lebesgue Integration, as in Probability and Measure, Part II Basic Functional Analysis (Hahn-Banach Theorem, Banach spaces)

Literature

1. Evans and Gariepy *Measure theory and Fine Properties of Functions* CRC Press 1992.
2. *Lecture Notes* by James Kilbane https://www.dpmms.cam.ac.uk/ jk511/ Also available at

   `https://www.dpmms.cam.ac.uk/~jk511/`

Combinatorics

Combinatorics (M16)

B. Bollobás

This course can be viewed as a continuation of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

In the extremal part of the course, we shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of 'classical' extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning such topics as 'concentration of measure' and hereditary properties of hypergraphs. There will be several indications of open problems.

Much of the course will be on the following material.

Extremal Combinatorics

Antichains; Sperner's lemma and related results. Shadows; compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Combinatorial Probability

Harris's Lemma, the van den Berg–Kesten Inequality and the Four Functions Theorem. The KKL Inequality and the Friedgut–Kalai Sharp Threshold Theorem.

Random Graphs

The basic models. Small subgraphs. The component structure. Connectedness. The phase transition.

Prerequisites

The basic concepts of graph theory and probability theory, and mathematical maturity.

Introductory Reading

Bollobás, B., Combinatorics, Set systems, hypergraphs, families of vectors and combinatorial probability, C.U.P. 1998, xii+177 pp.

Further Reading

The material in the course is covered by (small parts of) the following books.

Bollobás, B., Random Graphs, Second edition, Cambridge Studies in Advanced Mathematics, **73**, Cambridge University Press, Cambridge, 2001, xviii + 498 pp.

Bollobás, B., and Riordan, O., Percolation, C.U.P. 2006. x + 323 pp.

Frieze, A., and Karoński, M., Introduction to Random Graphs, C.U.P., 2015, xiii + 465 pp. by

Introduction to additive combinatorics (M16)

W. T. Gowers

This course will be an introduction to some of the most important theorems and proof techniques in additive combinatorics. Among the theorems covered will be the following.

1. Roth's theorem on arithmetic progressions states that for every $\delta > 0$ there exists a positive integer n such that every subset $A \subset \{1, 2, \ldots, n\}$ of size at least δn contains an arithmetic progression of length 3.

2. The Freiman-Ruzsa theorem gives a complete description of sets of integers A such that the sumset $A + A$ is not much bigger than A.

3. The cap-set problem, solved in May 2016, asks whether there is a constant $c < 3$ such that every subset of $\mathbb{F}_3^n$ of size at least c^n contains distinct elements x, y, z such that $x + y + z = 0$. (The answer is now known to be yes.)

4. Szemerédi's regularity lemma shows that all dense graphs can be decomposed into a bounded number of pieces, each of which is "quasirandom" in a sense that can be made precise.

The tools that will be explained will include the use of discrete Fourier analysis, use of regularity methods, exploiting the Cauchy-Schwarz inequality, and proving combinatorial theorems by considering zeros of multivariable polynomials. If time permits, I may discuss aspects of "higher-degree Fourier analysis" and its role in Szemerédi's theorem, which is the obvious generalization of Roth's theorem to longer arithmetic progressions.

Pre-requisites

A good understanding of the pure mathematics courses up to IB level (that is, up to the level of a second-year Cambridge undergraduate) is probably sufficient for this course.

Literature

The book Additive Combinatorics, by Terence Tao and Van Vu, is a very comprehensive introduction to additive combinatorics. Only a small fraction of the book will be directly relevant, but it is worth looking at to get an idea of where one can go next with the ideas covered in the course.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Ramsey Theory (L16)

B. P. Narayanan

What happens when we cut up a mathematical structure into many 'pieces'? How big must the original structure be in order to guarantee that at least one of the pieces has a specific property of interest? These are the kinds of questions at the heart of Ramsey theory. A prototypical result in the area is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes, there is a class that contains arbitrarily long arithmetic progressions.

The course will cover both classical material and more recent developments in the subject. Some of the classical results that I shall cover include Ramsey's theorem, van der Waerden's theorem and the Hales–Jewett theorem; I shall discuss some applications of these results as well. More recent developments that I hope to cover include the properties of non-Ramsey graphs, topics in geometric Ramsey theory, and finally, connections between Ramsey theory and topological dynamics. I will also indicate a number of open problems.

Pre-requisites

There are almost no pre-requisites and the material presented in this course will, by and large, be self-contained. However, students familiar with the basic notions of graph theory and point-set topology are bound to find the course easier.

Books

Many of the classical results in Ramsey theory may be found in the following books.

1. *Combinatorics*, B. Bollobás, Cambridge University Press 1986.
2. *Ramsey Theory*, R. Graham, B. Rothschild and J. Spencer, John Wiley 1990.

Topics in Random Graphs (L16)

Non-Examinable (Graduate Level)

Richard Montgomery

Random Graphs have been a fundamental object of study in combinatorics since the pioneering work of Erdős and Rényi in the 1960's. Not only do random graphs have many interesting properties, but their study provides a rich space in which to mix combinatorial and probabilistic ideas. In this course we will explore some of these ideas.

For any given (monotone) graph property (such as being connected), if we steadily increase the density of edges in a random graph then at a certain point - known as the threshold for the property - the random graph will suddenly become very likely to have this property. We will determine the thresholds of a range of different graph properties, and also prove various hitting time results, where a complex property is shown to almost surely hold in a random graph if (and only if) some other simpler property holds. The material covered will include coupling probability models, sprinkling edges, conditioning arguments, Pósa's rotation technique, absorption methods, and other techniques.

Pre-requisites

We will assume only some very basic notions of probability and graph theory.

Literature

1. S. Janson, T. Łuczak and A. Ruciński. *Random Graphs.* Wiley-Interscience, 2000.
2. B. Bollobás. *Random Graphs.* Cambridge University Press, 2001.
3. A. Frieze and M Karoński. *Introduction to Random Graphs.* Cambridge University Press, 2015.

Nilpotent additive combinatorics (L16)

Non-Examinable (Graduate Level)

Matthew Tointon

The field of additive combinatorics has been subject to a large volume of research in recent years, both because of its inherent interest and because of its many applications to other fields such as geometric group theory, number theory and differential geometry.

Additive combinatorics is broadly concerned with how finite subsets of groups behave when we add or multiply them together, and in particular how the algebraic structure of those sets governs this behaviour. For example, if A is a random set of size n inside the interval $\{1, 2, \ldots, n^{10}\}$ then we expect the set $A + A = \{a + a' : a, a' \in A\}$ to have size roughly quadratic in $|A|$. On the other hand, if A is highly structured then $|A + A|$ can be much smaller than this; for example, if A is a finite arithmetic progression then it is easy to see that $|A + A| \leq 2|A|$.

A classical theorem of Freiman asserts that an arithmetic progression exhibits precisely the kind of structure that prevents $|A + A|$ from being much larger than A. More explicitly, Freiman's theorem states that if $K > 1$ and if $A \subset \mathbf{Z}$ satisfies $|A + A| \le K|A|$ then A must essentially be a sum of at most $f(K)$ arithmetic progressions (where f is some explicit function).

In the past few years there has been considerable effort to generalise Freiman's theorem to groups other than the integers. The overall aim of this course is to present some of the results and techniques that have emerged from these generalisations. In particular, I will aim to develop the general machinery of so-called *approximate groups* in a way that will equip people attending the course to attack problems that are beyond the scope of these sixteen lectures.

The main result of the course will be a generalisation of Freiman's theorem to nilpotent groups, where it holds with a statement strongly analogous to that in the integers. Once one leaves the nilpotent setting new phenomena begin to emerge, and I will illustrate these phenomena by extending Freiman's theorem further, to residually nilpotent groups. I will also devote some time to applications in geometric group theory. In particular, I will prove (a strengthening of) Gromov's polynomial-growth theorem in the setting of residually nilpotent groups.

Pre-requisites

This course can in a sense be viewed as a sequel to the course *Introduction to additive combinatorics*, and for a well-rounded view of the subject it would certainly be preferable to have attended that course first. Nonetheless, this course will be sufficiently self contained that not having attended *Introduction to additive combinatorics* should not prevent anyone from understanding the material.

Literature

The course will be essentially self contained, but the following surveys give a high-level overview of the field and some of its applications.

1. E. Breuillard, *Lectures on approximate groups and Hilbert's 5th problem*, Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications **159** (2016), 369-404. Also available at `http://arxiv.org/abs/1512.01369`.

2. B. J. Green, *Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak*, Current events bulletin of the AMS (2010). Also available at `http://arxiv.org/abs/0911.3354`.

Hypergraph Games (E8)

Non-Examinable (Graduate Level)

Prof. I. B. Leader

Many natural games may be modelled as follows. We have a board (an arbitrary finite set), and some of its subsets are designated as 'winning'. Two players take it in turn to play on an (unoccupied) place on the board, and the first player to complete a winning subset is declared the winner. If when the board is full no player has won then the game is a draw.

The above is called the 'strong' version of the game. In the 'weak' version, also called 'maker-breaker', the second player's aim is not to occupy a winning set but just to prevent the first player from doing so. The interest is both for general theorems about games and also in particular games of interest, like the Hales-Jewett game (multidimensional noughts-and-crosses). Roughly speaking, a fair amount is known for maker-breaker while nothing at all is known for strong games.

The aim of the course is to give an introduction to this area. There are some very elegant and appealing results, and there are also many open problems.

We hope to cover the following material. **General games**

Basic examples. Weight functions; the Erdős-Selfridge theorem. The local lemma for games. Biased games.

The Hales-Jewett game

Pairing strategies; the Pairing Conjecture and the Ratio Conjecture. Beck's big game / little game approach.

Prerequisites

It would be helpful to have a basic knowledge of the Lóvasz Local Lemma, but this is by no means essential. It would be of similar use to have met the Hales-Jewett Theorem.

Geometry and Topology

Algebraic geometry (M24)

Caucher Birkar

This course is intended to serve as an introduction to modern algebraic geometry. One may define algebraic geometry as the study of solutions of systems of polynomial equations. This study is carried out by employing geometric intuition for guidance and advanced algebraic techniques for precision. Methods of algebraic geometry are so fruitful that they have found applications to subjects far beyond algebraic geometry such as number theory, differential geometry, analysis, topology, physics, logic, cryptography, etc.

Topics I hope to cover: sheaves, schemes, varieties, coherent sheaves, divisors, differential forms, cohomology, duality, Riemann-Roch theorem, etc.

Pre-requisites

Commutative algebra will be used throughout the course although most of it would be elementary. Some supplementary commutative algebra lectures might be offered.

Previous familiarity with algebraic geometry is not necessary but would be helpful.

Literature

[AM] M. Atiyah, I. Macdonald. *Introduction to commutative algebra.* Westview Press, 1994.

[H] R. Hartshorne. *Algebraic geometry.* Springer, 1977. (Much of the course is based on chapters II-III of this book.)

[S] I. Shafarevich. *Basic algebraic geometry I.* Springer, 1994.

Additional support

Four examples sheets will be provided and two associated examples classes will be given.

Algebraic Topology (M24)

O. Randal-Williams

Algebraic Topology assigns algebraic invariants to topological spaces; it permeates modern pure mathematics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. The course will not assume any knowledge of Algebraic Topology, but will go quite fast in order to reach more interesting material, so some previous exposure to simplicial homology or the fundamental group would be helpful. The Part III Differential Geometry course will also contain useful, relevant material.

Hatcher's book is especially recommended for the course, but there are many other suitable texts.

Literature

1. Hatcher, A. *Algebraic Topology.* Cambridge Univ. Press, 2002.
2. May, P. *A concise course in algebraic topology.* Univ. of Chicago Press, 1999.
3. Sutherland, W. *Introduction to metric and topological spaces.* Oxford Univ. Press, 1999.

The course will emphasise examples and computations; it will be accompanied by four question sheets with associated Examples Classes, which will again involve applying the general theory to do explicit calculations and solve geometric problems. There will be a one-hour revision class in the Easter term.

Differential Geometry (M24)

J. Ross

This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, smooth maps. Tangent vectors and vector fields, tangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Lie Groups.
- *Vector Bundles.* Structure group. The example of Hopf bundle. Bundle morphisms and automorphisms. Exterior algebra of differential forms. Tensors. Symplectic forms. Orientability of manifolds. Partitions of unity and integration on manifolds, Stokes Theorem; de Rham cohomology. Lie derivative of tensors. Connections on vector bundles and covariant derivatives: covariant exterior derivative, curvature. Bianchi identity.
- *Riemannian Geometry.* Connections on the tangent bundle, torsion. Bianchi's identities for torsion free connections. Riemannian metrics, Levi–Civita connection, Christoffel symbols, geodesics. Riemannian curvature tensor and its symmetries, second Bianchi identity, sectional curvatures.

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor's theorem in several variables). Exposure to some of the ideas of classical differential geometry might also be useful.

Literature

The most relevant sources are the two books by Lee listed below, as the course will likely follow parts of these texts.

1. J. Lee *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics, Springer-Verlag, 2002.
2. J. Lee *Riemannian Manifolds. An Introduction to Curvature* Graduate Texts in Mathematics, Springer-Verlag, 1997.
3. M. Spivak, *A Comprehensive Introduction to Differential Geometry, Volume 1.* Publish or Perish, 1999.
4. F.W. Warner, *Foundations of differentiable manifolds and Lie groups.* Graduate Texts in Mathematics, Springer-Verlag, 1983.

Additional support

Three or four examples sheets will be provided and four associated examples classes will be given. The fourth class will take place at the start of the Lent Term and will also fulfil a revision function.

Riemannian Geometry (L24)

Alexei Kovalev

This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:

A closer look at geodesics and curvature. Brief review from the Differential Geometry course. Geodesic coordinates and Gauss' lemma. Jacobi fields, completeness and the Hopf–Rinow theorem. Variations of energy, Bonnet–Myers diameter theorem and Synge's theorem.

Hodge theory and Riemannian holonomy. The Hodge star and Laplace–Beltrami operator. The Hodge decomposition theorem (with the 'geometry part' of the proof). Bochner–Weitzenböck formulae. Holonomy groups. Interplays with curvature and de Rham cohomology.

Ricci curvature. Fundamental groups and Ricci curvature. The Cheeger–Gromoll splitting theorem.

Additional support

The lectures will be supplemented by three example classes, the last class to include revision.

Pre-requisite Mathematics

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course *Differential Geometry* offered in Michaelmas Term is the ideal pre-requisite.

Literature

1. S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.
2. M.P. do Carmo, *Riemannian geometry.* Birkhäuser, 1993.
3. I. Chavel, *Riemannian geometry, a modern introduction*, CUP 1995.
4. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983. Chapter 6.
5. P. Petersen, *Riemannian geometry*, Springer-Verlag, 1998.
6. A.L. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

The first few chapters of do Carmo's text provide a good introductory reading.

Riemann surfaces and Teichmüller theory (L24)

Stergios M. Antonakoudis

This is an introduction to the theory of conformal dynamical systems, Riemann surfaces and their moduli spaces. One of the main goals of this course will be to construct and study the universal family of Riemann surfaces, using tools from complex analysis, geometry & dynamics. Depending on time, I hope to discuss (at least one) application to algebraic geometry (Shafarevic conjecture), topology (geometrization of 3-manifolds) or dynamics (billiards on polygons).

In the first part of the course, we will give a review of basic material on Riemann surfaces and discuss several examples of geometric structures in detail to motivate the rest of the discussion.

Likely topics include: Extremal length and quasiconformal mappings; Teichmüller's theorem. Quadratic differentials and the $SL(2,\mathbb{R})$ action on their moduli space; applications to the dynamics of billiards on polygons. Poincaré uniformization, Fuchsian groups and hyperbolic geometry of Riemann surfaces; Bers' simultaneous uniformization and quasi-Fuchsian groups. Holomorphic motions, analytic families of Riemann surfaces and the universal property of Teichmüller space. The infinitesimal shape of its Kobayashi metric and Royden's theorem; holomorphic and isometric rigidity, dynamics of Kleinian groups and the Shafarevich conjecture.

Pre-requisites

We will assume familiarity with basic Complex Analysis, Geometry (metrics & geodesics on surfaces, the hyperbolic plane) and Algebraic Topology (fundamental groups, covering spaces).

Part IB Geometry and Complex Analysis & Part II Differential Geometry, Riemann Surfaces and Algebraic Topology (or their equivalent) are essential for this course. Part II Algebraic Geometry, Linear Analysis & Part III Differential Geometry and Algebraic Topology are useful.

Literature

1. J. H. Hubbard, *Teichmller Theory, vol. 1.* Matrix Editions, 2006
2. S. Nag, *The Complex Analytic Theory of Teichmüller Space*, Wiley, 1988
3. O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987
4. L. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw, 1973
5. C. T. McMullen, *Complex Analysis on Riemann surfaces*, Course Notes `http://math.harvard.edu/~ctm/papers/index.html#books`
6. C. T. McMullen, *Riemann surfaces, dynamics and geometry*, Course Notes `http://math.harvard.edu/~ctm/papers/index.html#books`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Geometric group theory (L16)

Mark Hagen

Geometric group theory is the study of algebraic and algorithmic properties of infinite groups via their actions on spaces. The universe of groups is hopelessly complicated, but there is a precise sense in which a "generic" finitely presented group is a tractable object, because it exhibits "negative curvature". We will examine such "negatively-curved" – hyperbolic – groups. To motivate the definition of hyperbolicity, to make concrete the connection with familiar notions of curvature, and to trace the historical roots, much of the course will be devoted to combinatorial curvature and small-cancellation theory.

Part 1. *Groups as geometric objects (4 lectures).* Free groups, presentations, presentation complexes, Cayley graphs. Geodesic metric spaces, quasi-isometries. The Milnor-Švarc lemma. Examples and motivation: graphs of groups, algorithmic problems.

Part 2. *Combinatorial negative curvature (5-7 lectures).* Angled 2–complexes, curvature and the combinatorial Gauss-Bonnet theorem. Disc diagrams, van Kampen lemma. Small-cancellation conditions. Structure of disc diagrams under small-cancellation, Greendlinger lemma. Group-theoretic consequences of small-cancellation. Strebel's classification of geodesic triangles.

Part 3. *Hyperbolic groups (7-5 lectures).* Definition of hyperbolicity and relationship with small-cancellation. Linear isoperimetric inequality. Non-examples. Random groups. The Rips construction. The Gromov boundary, ping-pong, and free subgroups of hyperbolic groups. Quasi-isometry invariant properties of hyperbolic groups. More applications of small-cancellation theory: Tarski monsters, continuously many QI types of 2-generator groups,...

Pre-requisites

Part II Algebraic Topology and Part IB Geometry.

Literature

The following supplemental literature may be useful; the main resource will be the course notes.

1. S.M. Gersten. *Introduction to hyperbolic and automatic groups.* Summer School in Group Theory in Banff, 1996. `http://www.math.utah.edu/~sg/Papers/banff.pdf`
2. R. Strebel. *Small cancellation groups.* Appendix to *Sur les groupes hyperboliques d'après Gromov*, Birkhäuser, 1990.
3. J.P. McCammond and D.T. Wise. *Fans and ladders in small cancellation theory.* Proceedings of the London Mathematical Society 84.3 (2002): 599-644.
4. M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Linear Systems (L16)

Roberto Svaldi

This class aims at giving an introduction to the theory of divisors and linear systems on projective algebraic varieties.

The first part of the class will be dedicated to introducing the basic notions and results regarding these objects and special attention will be devoted to discussing examples in the case of curves and surfaces.

In the second part, the course will cover classical results from the theory of divisors and linear systems and their applications to the study of the geometry of algebraic varieties.

If time allows and based on the interests of the participants, there are a number of more advanced topics that could possibly be covered: Reider's Theorem for surfaces, geometry of linear systems on higher dimensional varieties, multiplier ideal sheaves and invariance of plurigenera, higher dimensional birational geometry.

Pre-requisites

The minimum requirement for those students wishing to enroll in this class is their knowledge of basic concepts from the Algebraic Geometry Part 3 course, i.e., roughly Chapters 2 and 3 of Hartshorne's Algebraic Geometry.

Familiarity with the basic concepts of the geometry of algebraic varieties of dimension 1 and 2 - e.g., as covered in the preliminary sections of Chapters 4 and 5 of Hartshorne's Algebraic Geometry - would be useful but will not be assumed.

Students should have also some familiarity with concepts covered in the Algebraic Topology Part 3 course such as cohomology, duality and characteristic classes.

Literature

1. W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces.* Springer, 1984.
2. R. Hartshorne, *Algebraic Geometry.* Springer, 1997.
3. J. Kollár, S. Mori, *Birational geometry of algebraic varieties.* Cambridge University Press, 1998.
4. R. Lazarsfeld, *Positivity in Algebraic Geometry, Vol. 1.* Springer, 2004.
5. D. Mumford, *Lectures on Curves on an Algebraic Surface.* Princeton University Press, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Algebraic Geometry (L16)

Non-Examinable (Graduate Level)

M. Gross

The course will be devoted to various aspects of Gromov-Witten invariants, which are curve-counting invariants of, in the context of this course, non-singular algebraic varieties. I will begin by defining the Kontsevich stable map moduli space and explain how the virtual fundamental class on this moduli space is constructed. Elementary properties of the invariants will be proved, leading to the construction of the quantum cohomology ring, a perturbation of the usual cup product on cohomology. Depending on time, I will then cover a number of calculational techniques and perhaps discuss logarithmic Gromov-Witten invariants.

Pre-requisites

The minimal prerequisite is the Part III course in Algebraic Geometry. Basics of Algebraic Topology will also be helpful.

Literature

1. W. Fulton, R. Pandharipande, *Notes on stable maps and quantum cohomology*, arXiv:alg-geom/9608011.
2. D. Cox, S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999. xxii+469 pp.
3. K. Behrend, B. Fantechi, *The intrinsic normal cone*, arXiv:alg-geom/9601010.

Stein manifolds and symplectic cohomology (L16)

Non-Examinable (Graduate Level)

Ivan Smith

A Stein manifold is a closed holomorphic submanifold of complex Euclidean space; an important special class of Stein manifolds is the class of affine algebraic varieties. This will be a topics course on symplectic topological features of Stein manifolds, in part as illuminated by symplectic cohomology, a version of Floer theory well-adapted to their study. This is a large subject, and a full development of symplectic cohomology has heavy analytic foundations, so most proofs will only be sketched, in the hope of giving some geometric intuition and an overview of the subject, with an emphasis on open problems.

Likely topics include: Weinstein handles and topology of Stein manifolds; Milnor fibres and isolated singularities; exotic Stein structures on Euclidean space; symplectic cohomology of cotangent bundles and undecidability questions; growth rates and topological obstructions to being affine algebraic.

Pre-requisites

We will assume familiarity with basic symplectic and contact topology; some exposure to holomorphic curve theory would be useful, but not essential.

Literature

1. P. Seidel. *A biased view of symplectic cohomology.* Available at arXiv:0704.2055.

Logic

Category Theory (M24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

Locally small categories. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

Limits. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Regular categories. Embedding theorems. Categories of relations, introduction to allegories.

Abelian categories. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.
4. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Logic (L24)

Thomas Forster

This course is the sequel to the Part II courses in *Set Theory and Logic* and in *Automata and Formal Languages* lectured in 2015-6. (It is already being referred to informally as "Son of ST& L and Automata & Formal Languages"). Because of the advent of that second course this Part III course no longer covers elementary computability in the way that its predecessor ("Computability and Logic") did, and this is reflected in the change in title. It will say less about Set Theory than one would expect from a course entitled 'Logic'; this is because in Lent term Benedikt Löwe will be lecturing a course entitled 'Topics in Set Theory' and I do not wish to tread on his toes. Material likely to be covered include: advanced topics in first-order logic (Natural Deduction, Sequent Calculus, Cut-elimination, Interpolation, Skolemisation, Completeness and Undecidability of First-Order Logic, Curry-Howard, Possible world semantics, Gödel's Negative Interpretation, Generalised quantifiers . . .); Advanced Computability (λ-representability of computable functions, Tennenbaum's theorem, Friedberg-Muchnik, Baker-Gill-Solovay. . .); Model theory background (ultraproducts, Łoś's theorem, elementary embeddings, omitting types, categoricity, saturation, Ehrenfeucht-Mostowski theorem. . .); Logical combinatorics (Paris-Harrington, WQO and BQO theory at least as far as Kruskal's theorem on wellquasiorderings of trees. . .).

This is a new syllabus and may change in the coming months. It is entirely in order for students to contact the lecturer for updates.

Pre-requisite Mathematics

The obvious prerequisites from last year's Part II are Professor Johnstone's *Set Theory and Logic* and Dr Chiodo's *Automata and Formal Languages*, and I would like to assume that everybody coming to my lectures is on top of all the material lectured in those courses. This aspiration is less unreasonable than it may sound, since in 2016-7 both these courses are being lectured the term before this one, in Michaelmas; indeed supervisions for Part III students attending them can be arranged if needed: contact me or your director of studies. I am lecturing Part II Set Theory and Logic and I am even going to be issuing a "Sheet 5" for Set Theory and Logic, of material likely to be of interest to people who are thinking of pursuing this material at Part III. Attending these two Part II courses in Michaelmas is a course of action that may appeal particularly to students from outside Cambridge.

Literature

J.L. Bell and Alan Slomson Models and Ultraproducts. Dover

T. E. Forster: Logic, Induction and Sets CUP.
(Errata are on `http://www.dpmms.cam.ac.uk/~tf/typoslis.html`)

P. T. Johnstone: Notes on Set theory CUP.

Wilfrid Hodges: Model theory CUP (long and short versions).

Eliot Mendelson: Introduction to Mathematical Logic.

Teaching materials will be linked from the page on `http://www.dpmms.cam.ac.uk/~tf/partiii.html`

Topics in Set Theory (L24)

B Löwe

This course covers advanced topics in set theory, focusing on meta-mathematical techniques such as inner models and forcing.

Set theory and logic are intrinsically intertwined since the most interesting results in set theory are independence results showing that natural questions in set theory are not solvable using the standard axiomatic system of Zermelo-Fraenkel set theory with choice ZFC.

The most famous of these natural questions is Cantor's continuum hypothesis CH, "every uncountable set of reals is equinumerous to the set of all real numbers" or, equivalently, $2^{\aleph_0} = \aleph_1$. This question was elevated to the status of the foremost mathematical problem for the 20th century by David Hilbert in his address to the *International Congress of Mathematicians* in Paris in the year 1900. In 1938, Kurt Gödel proved that CH cannot be disproved in ZFC (inventing and using the method of inner models); in 1963, Paul Cohen proved that CH cannot be proved in ZFC (inventing and using the method of forcing). Together, these results show that CH is independent from ZFC.

We shall treat several of the following topics:

Model theory of set theory. Models of set theory. Absoluteness. Simple independence results. Ranks. Reflection principles.

Inner models. Definability. Ordinal definability. Constructibility. Condensation. Gödel's proof of the consistency of CH.

Large cardinals. Introduction to large cardinals. Inaccessible cardinals. Measurable cardinals. Ultrapowers. Scott's theorem.

Forcing. Generic extensions. The forcing theorems. Adding reals; collapsing cardinals. Cohen's proof of the consistency of $\neg$CH.

Pre-requisites

The Part II course *Logic and Set Theory* or an equivalent course is essential.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Number Theory

Local Fields (M24)

Christian Johansson

The p-adic numbers $\mathbb{Q}_p$ (where p is any prime) were invented by Hensel in the late 19th century, with a view to introduce function-theoretic methods into number theory. They are formed by completing $\mathbb{Q}$ with respect to the p-adic absolute value $|-|_p$, defined for non-zero $x \in \mathbb{Q}$ by $|x|_p = p^{-n}$, where $x = p^n a/b$ with $a, b, n \in \mathbb{Z}$ and a and b are coprime to p. The p-adic absolute value allows one to study congruences modulo all powers of p simultaneously, using analytic methods. The concept of a local field is an abstraction of the field $\mathbb{Q}_p$, and the theory involves an interesting blend of algebra and analysis. Local fields provide a natural tool to attack many number-theoretic problems, and they are ubiquitous in modern algebraic number theory and arithmetic geometry.

Topics likely to be covered include:

The p-adic numbers. Local fields and their structure.

Finite extensions, Galois theory and basic ramification theory.

Polynomial equations; Hensel's Lemma, Newton polygons.

Continuous functions on the p-adic integers, Mahler's Theorem.

Local class field theory (time permitting).

Pre-requisites

Basic algebra, including Galois theory, and basic concepts from point set topology and metric spaces. Some prior exposure to number fields might be useful, but is not essential.

Literature

The books by Neukirch and Serre cover most of the material that will be included in the course (and much more), with the exception of Mahler's Theorem (which is covered in Robert's book).

1. J. P. Serre, *Local Fields*, Springer, 1979.

2. J. Neukirch, *Algebraic Number Theory*, Springer, 1999.

3. A. Robert, *A Course in p-adic analysis*, Springer, 2000.

Modular Forms and L-Functions (L24)

Prof. A. J. Scholl

Modular Forms are classical objects that appear in many areas of mathematics, from number theory to representation theory and mathematical physics. Most famous is, of course, the role they played in the proof of Fermat's Last Theorem, through the conjecture of Shimura-Taniyama-Weil that elliptic curves are modular. One connection between modular forms and arithmetic is through the medium of L-functions, the basic example of which is the Riemann ζ-function. We will discuss various types of L-function in this course and give arithmetic applications.

Pre-requisite Mathematics

Prerequisites for the course are fairly modest; from number theory, apart from basic elementary notions, some knowledge of quadratic fields is desirable. A fair chunk of the course will involve (fairly 19th-century) analysis, so we will assume the basic theory of holomorphic functions in one complex variable, such as are found in a first course on complex analysis (e.g. the 2nd year Complex Analysis course of the Tripos).

Level: Basic

Books

1. J. P. Serre, *A course in Arithmetic*, Graduate Texts in Maths. **7**, Springer, New York, 1973 (Chapter VII is an easy-going introduction to modular forms, and Chapter VI covers Dirichlet L-functions and the theorem on primes in arithmetic progresions.).

2. D. Bump, *Automorphic forms and representations*, Cambridge Studies in Adv. Maths. **55**, CUP, Cambridge, 1997 (Sections 1.1-1.6 of Chapter I are particularly relevant).

3. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. **228**, Springer, New York, 2005 (a good reference providing also an introduction to the algebraic theory of modular forms, although goes into a lot more detail than we will give in this course).

4. J. Milne, *Modular Functions and Modular Forms*, Lecture notes from a course, download available at `http://www.jmilne.org/math`.

Additional references for enthusiasts

5. T. Miyake, *Modular Forms*, Springer, Berlin, 1989 (a standard reference for classical theory of modular forms).

6. F. Diamond, J. Im, *Modular forms and modular curves*, in: *Seminar on Fermat's Last Theorem*, CMS Conf. Proc. 17, Amer. Math. Soc., Providence, RI, 1995, 39-133.

7. J. Coates, Shing-Tung Yau, *Elliptic curves, modular forms & Fermat's last theorem- Conference Proceedings*, International Press, Cambridge, MA, 1997 (in particular, the survey article by H. Darmon, F. Diamond, R. Taylor).

8. H. Hida, *Elementary theory of L-functions and Eisenstein series*, London Math. Soc. Student Texts **26**, CUP, Cambridge, 1993 (not so elementary introduction to arithmetic of modular forms).

Elliptic Curves (L24)

Tom Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The following topics will be covered, and possibly others if time is available.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process. Associativity via the identification with the Jacobian. Elliptic curves as group varieties.

Isogenies. Definition and examples. The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Pre-requisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a review of the necessary geometric background, but some previous knowledge of algebraic curves (at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]) would be very helpful. Later in the course, some basic knowledge of the field of p-adic numbers will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, Rational Points on Elliptic Curves, Springer, 1992.

Literature

2. J.W.S. Cassels, Lectures on Elliptic Curves, CUP, 1991.
3. J.H. Silverman, The Arithmetic of Elliptic Curves, Springer, 1986.

Additional support

There will be four example sheets and four associated examples classes.

Topics in number theory (L16)

Non-Examinable (Graduate Level)

Jack Thorne

The aim of this graduate level course will be to develop the relation between the arithmetic of elliptic curves and the invariant theory of binary quartic forms, and to understand its recent applications to the statistics of ranks of elliptic curves over global fields. This relation was exploited by Birch and Swinnerton-Dyer in their first wide-ranging calculations of ranks of elliptic curves, and is used in Cremona's famous `mwrank` program for calculating ranks.

In a recent landmark work, Bhargava and Shankar have applied new orbit-counting techniques to binary quartic forms in order to show that the average rank of an elliptic curve over the field of rational numbers is bounded above. This course will be devoted to the techniques involved in the proof of this result and, time permitting, its generalizations.

Pre-requisites

We will assume background knowledge at the level of Part II Algebraic Geometry and Part III Algebraic Number Theory, but some parts of the course will go beyond this. You may find it helpful to attend the Part III Elliptic Curves course at the same time.

Literature

1. Birch, B. J. and Swinnerton-Dyer, H. P. F. *Notes on elliptic curves. I.* J. Reine Angew. Math. 212 (1963), pp. 7–25
2. Cremona, J. E. and Fisher, T. A. *On the equivalence of binary quartics.* J. Symbolic Comput. 44 (2009), no. 6, pp. 673–682.
3. Bhargava, M. and Shankar, A. *Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves.* Ann. of Math. (2) 181 (2015), no. 1, 191–242.
4. Ho, Q. P. and Lê Hùng, V. B. and Ngô, B. C., *Average size of 2-Selmer groups of elliptic curves over function fields.* Math. Res. Lett. 21 (2014), no. 6, pp. 1305–1339.

Probability and Finance

Advanced Probability (M24)

James Norris

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

Review of measure and integration: sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.
Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
Poisson random measures: Construction and properties; integrals.
Lévy processes: Lévy-Khinchin theorem.

Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- Lecture notes online: `www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`
- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with supervisions. There will be a revision class in Easter term.

Percolation and Random Walks on Graphs (M16)

Perla Sousi

A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm-Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

1. Bollobás, B. and Riordan, O., *Percolation*, Cambridge University Press, 2006
2. Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010 available at `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`
3. Grimmett, G. R., *Percolation*, Springer-Verlag, Berlin, second edition, 1999.
4. Lyon, R. and Peres, Y., *Probability on Trees and Networks* available at `http://mypage.iu.edu/~rdlyons/prbtree/book_online.pdf`

Mixing Times of Markov Chains (M16)

Nathanaël Berestycki

Elementary theory tells us that, after a sufficiently large time, the distribution of an irreducible, aperiodic, finite Markov chain is close to the invariant distribution. But how long should one wait in practice? For instance, how many times should a deck of cards be shuffled before its distribution is approximately uniform? This type of questions is at the heart of the theory of mixing times of Markov chains. It is a surprisingly rich question, with ramifications in analysis, geometry, combinatorics, representation theory, etc.

We shall focus on the basic theory and expose some of the main techniques which have been used to tackle this question. Our main goal will be to discuss the *cutoff phenomenon*, which says that a Markov chain reaches its stationary distribution in an abrupt fashion, after a well-defined number of steps called the *mixing time*. Surprisingly this phenomenon seems to be widespread.

A rough plan of the course is as follows:

Coupling method: convergence in total variation distance.

Spectral methods: eigenvalue decomposition and relaxation time.

Geometric methods: canonical paths, Cheeger's inequality, expanders.

Analytic methods: comparison theorem of Diaconis and Saloff-Coste.

Other notions of stationarity: strong stationary times, cover times, Lovász–Winkler theory.

Pre-requisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level.

Literature

1. D. Levin, Y. Peres and E. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.
2. N. Berestycki. *Mixing Times of Markov Chains: Techniques and Examples.* Available on the webpage of the author.
3. R. Montenegro and P. Tetali. *Mathematical aspects of mixing times in Markov chains.* Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3, pp 237–354, 2006.

Additional support

Examples sheets will be provided and associated examples classes will be given.

Stochastic Calculus and Applications (L24)

V. Silvestri

This course will be an introduction to Itô calculus.

- *Brownian motion.* Existence and sample path properties.
- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins-Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, Yamada-Watanabe theorem, strong Markov property, and relation to second order partial differential equations.
- *Stroock–Varadhan theory.* Diffusions, martingale problems, equivalence with SDEs, approximations of diffusions by Markov chains.

Pre-requisites

We will assume knowledge of measure theoretic probability as taught in Part III Advanced Probability. In particular we assume familiarity with discrete-time martingales and Brownian motion.

Literature

1. R. Durrett *Probability: theory and examples.* Cambridge. 2010
2. I. Karatzas and S. Shreve *Brownian Motion and Stochastic Calculus.* Springer. 1998
3. P. Morters and Y. Peres *Brownian Motion.* Cambridge. 2010
4. D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* Springer. 1999
5. L.C. Rogers and D. Williams *Diffusions, Markov Processes, and Martingales.* Cambridge. 2000

Schramm-Loewner Evolutions (L16)

J. P. Miller

Schramm-Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Literature

1. Nathanaël Berestycki and James Norris. Lecture notes on SLE.
   `http://www.statslab.cam.ac.uk/~james/Lectures`
2. Wendelin Werner. *Random planar curves and Schramm-Loewner evolutions*,
   arXiv:math.PR/0303354, 2003.
3. Gregory F. Lawler. *Conformally Invariant Processes in the Plane*, AMS, 2005.

Additional support

Two examples sheets will be provided and examples classes given. There will be a revision class in Easter Term.

Statistics and Operational Research

The courses in statistics form a coherent Masters-level course in statistics, covering statistical methodology, theory and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
First	*Essential*	Probability
Second	*Essential*	Statistics
	Helpful for some courses	Markov Chains
Third	*Helpful*	Principles of Statistics
	Helpful for applied statistics courses	Statistical Modelling
	For additional background	Probability and Measure

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Modern Statistical Methods (M24)

Rajen Shah

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover a selection of the following topics.

- Penalised regression: Ridge regression, the Lasso and variants. High-dimensional inference, including the debiased Lasso.
- Machine learning methods: Boosting, Support vector machines, the kernel trick and the hashing trick. Generalisation error bounds and Rademacher complexity.

- Multiple testing: the closed testing procedure and the Benjamini–Hochberg procedure.
- Graphical modelling: neighbourhood selection and the graphical Lasso. Causal inference through structural equation modelling; the PC algorithm.

Pre-requisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.
3. T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations.* CRC Press, 2015.
4. C. Giraud, *Introduction to High-Dimensional Statistics.* CRC Press, 2014.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Topics in Statistical Theory (M16)

T. Cannings

The objective of this course is to give an introduction to topics in modern statistical theory. Important problems in nonparametric statistics will be covered, as well as some techniques used to solve them. Emphasis will be placed on theoretical results. We will aim to cover the following topics.

- Nonparametric statistics: density estimation, regression, adaptive inference.

- Minimax theory: notion of information-theoretic lower bounds, distance and divergence between distributions, optimal rates.

- Classification problems: the Bayes classifier, nearest neighbour methods.

Pre-requisites

A good background in probability theory, as well as elements of linear algebra and functional analysis. A preliminary course in mathematical statistics can be helpful, but it is not necessary.

Literature

No book will be explicitly followed, but some of the material is covered in

L. Devroye, L. Györfi, G. Lugosi, *A Probabilistic Theory of Pattern Recognition*, Springer 1996

A. Tsybakov, *Introduction to Nonparametric Estimation*, Springer 2009

Additional support

Three example sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Applied Statistics (Michaelmas and Lent (24))

Davide Pigoli

This course is split over two terms, with 16 hours (8 lectures and 8 practical classes) in the Michaelmas Term and 8 hours (4 lectures and 4 practical classes) in the Lent Term. It is a practical course aiming to develop skills in analysis and interpretation of data. The practical classes will deal with an introduction to R, exploratory data analysis (data visualization and dimensional reduction) and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models and quasi-likelihood methods.
- Non parametric regression, generalized additive models, introduction to functional data analysis.
- Mixed effects models and longitudinal data.
- Empirical and parametric bootstrap.
- Time series: spectral analysis, ARMA models and forecasting.
- Basics of spatial statistics.

Pre-requisites

Elementary probability theory. Maximum likelihood estimation, hypothesis tests and confidence intervals. Linear models.

Previous experience with R is helpful but not essential.

Literature

1. Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.
2. Efron, B. and Tibshirani, R. J. (1994) *An introduction to the bootstrap.* CRC press.
3. Faraway, J. J. (2005) *Extending the linear model with R: generalized linear, mixed effects and non-parametric regression models.* CRC press.
4. Gaetan, C. and Guyon, X. (2010). *Spatial statistics and modeling.* New York: Springer.
5. Shumway, R. H., and Stoffer, D. S. (2010) *Time series analysis and its applications: with R examples.* Springer Science & Business Media.
6. Wood, S. (2006) *Generalized additive models: an introduction with R.* CRC press.

Additional support

This course includes practical classes in both the Michaelmas and Lent Terms, where statistical methods are introduced in a practical context and where students carry out analysis of datasets using R. In practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four examples sheets will be provided and there will be four associated examples classes. There will be a revision class in the Easter Term.

Biostatistics (M10+L14)

This course consists of two components, Statistics in Medical Practice (10 lectures) and Analysis of Survival Data (14 lectures). Together these make up one 3 unit (24 lecture) course. You must take the two components together for the examination.

Statistics in Medical Practice (M10)

R. Turner, C. Jackson, I. White, D. de Angelis, A. Presanis, P. Birrell, P. Newcombe, S. Hill, P. Kirk

Each lecture will be a self-contained study of a topic in biostatistics, which may include clinical trials, meta-analysis, missing data, multi-state models, statistical genomics and infectious disease modelling. The relationship between the medical issue and the appropriate statistical theory will be illustrated.

Pre-requisites

Undergraduate-level statistical theory, including estimation, hypothesis testing and interpretation of findings.

Literature

It would be very useful to have some familiarity with media coverage of medical stories involving statistical issues, e.g. from Behind the Headlines on the NHS Choices website: `http://www.nhs.uk/News/Pages/NewsIndex.aspx`. A book to complement the course material is mentioned below.

1. Armitage P, Berry G, Matthews JNS. *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001

Additional support

A two-hour example class will be given, with question sheets and solutions.

Analysis of Survival Data (L14)

F. P. Treasure

Fundamentals of Survival Analysis:

Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

Current Topics in Survival Analysis:

In recent years there have been lectures on: frailty, cure, relative survival, empirical likelihood, counting processes and multiple events.

Pre-requisites

This course assumes assumes that you have attended an undergraduate course in statistics and that you are familiar with hypothesis testing, point and interval estimation, and likelihood methods. Attendance at Michaelmas Term Part III statistics courses would be helpful, but is not essential.

Literature

1. P. Armitage, J. N. S. Matthews and G. Berry *Statistical Methods in Medical Research* (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].

2. D. R. Cox and D. Oakes *Analysis of Survival Data* London: Chapman and Hall (1984).

3. M. K. B. Parmar and D. Machin *Survival Analysis: A Practical Approach* Chichester: John Wiley (1995)

4. Therneau T.M. and Grambsch P.M. *Modelling Survival Data: Extending the Cox Model* New York: Springer (2000)

Additional support

A two hour examples class will be provided in the Lent Term. There will be a two hour revision class based on student-selected examination questions in the Easter Term.

Bayesian Modelling and Computation (L24)

Sergio Bacallado

The course will cover a range of algorithms for sampling and numerical integration which are useful in Bayesian inference. A third of the lectures will deal with applications in specific statistical models.

- **Fundamentals:** Monte Carlo integration; variance reduction techniques; rejection sampling and adaptive rejection sampling; importance sampling. Exponential families and conjugate priors.
- **Graphical models:** Belief networks, Markov random fields, and factor graphs; Hammersley–Clifford theorem; computational reduction between marginalisation, computing partition function, and sampling; belief propagation in trees.
- **Algorithms:** Markov chain Monte Carlo, conditions for convergence. Metropolis–Hastings and pseudo-marginal variants. Hamiltonian Monte Carlo; symplectic reversible integrators. Gibbs sampling. Simulated annealing and parallel tempering. Sequential Monte Carlo.
- **Approximate inference:** Expectation maximisation. Variational inference; mean-field methods. Stochastic variational inference. The parametric Bootstrap and Bayesian methods.
- **Modelling examples:** Hidden Markov Models for time series. Bayesian generalised linear models; mixed-effects models; variable selection priors. Mixture models; block and collapsed Gibbs sampler; hierarchical mixtures. Feature models. Nonparametric priors and slice sampling. Gaussian process priors in spatio-temporal models.

Pre-requisites

This course assumes familiarity with probability and basic Markov chain theory. Knowledge of statistical modelling and Bayesian analysis is helpful.

Literature

1. Murphy, K., *Machine Learning: a Probabilistic Perspective,* MIT Press, 2012.

2. Brooks, S., Gelman, A., Jones, G., Meng, X-L. *Handbook of Markov Chain Monte Carlo,* 1st edition. Chapman and Hall, 2011.

3. Owen, A.B., *Monte Carlo theory, methods, and examples*, 2013. Available at

   `http://statweb.stanford.edu/~owen/mc/`

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Gaussian Processes (L16)

Richard Nickl

The study of Gaussian processes is concerned with infinite-dimensional notions of the 'normal', or 'Gaussian' distribution, and their properties. Gaussian processes are fundamental building blocks in several areas of modern mathematics, ranging from probability and functional analysis to statistics and learning theory. In this course we will give a rigorous account of some of the classical results about Gaussian processes, and illustrate their usefulness in several applications, particularly in statistics.

The course will roughly be structured as follows:

I. Basic definitions and properties of Gaussian processes with abstract index sets. Gaussian measures on function spaces as sample-continuous Gaussian processes. Concentration properties of norms of Gaussian processes, zero-one law, Fernique's theorem. Examples and applications.

II. Maximal inequalities, chaining, Dudley's metric entropy inequality for suprema of Gaussian processes, Slepian's comparison lemma, Sudakov's lower bound. Applications to empirical processes and high-dimensional statistics

III. Reproducing kernel Hilbert spaces, Bochner integrals, Karhunen-Loève series expansion of Gaussian measures and processes, Cameron-Martin theorem. Applications to statistical minimax theory and Bayesian non-parametric statistics

Pre-requisites

Backgound in probability & measure theory is necessary. Basic knowledge of linear & functional analysis will be useful; the book [2] covers all the pre-requisites (and much more).

The course will be loosely based on Chapter 2 of the book [3].

Further relevant (but advanced) references are [1] and [4].

Literature

1. V. L. Bogachev, *Gaussian measures*, AMS monographs, Providence 1998
2. R.M. Dudley, *Real analysis and probability*, Cambridge University Press, Cambridge 2002
3. E. Giné & R. Nickl, *Mathematical foundations of infinite-dimensional statistical models*, Cambridge University Press, Cambridge 2016
4. M. Talagrand, *Upper and lower bounds for stochastic processes,* Springer Heidelberg, 2014.

Additional support

Examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Nonparametric inference under shape constraints (M8)

Non-Examinable (Graduate Level)

Richard Samworth

Nonparametric shape constraints, such as monotonicity, convexity or log-concavity, offer statisticians the potential of freedom from restrictive parametric assumptions, while often still permitting fully automatic procedures (with no tuning parameters to choose!). In this sense, they combine the best of both the parametric and nonparametric worlds.

To fix ideas in a setting where explicit solutions are available, the course will start by studying the famous Grenander (1956) estimator, namely the nonparametric maximum likelihood estimator of a decreasing density on the positive half-line. The rest of the course, however, will be largely focused on research carried out over the last 6-7 years. A particular emphasis will be on the beautiful class of log-concave densities on $\mathbb{R}^d$, and how we might estimate them.

Other topics, including shape-constrained regression problems and semiparametric inference will also be treated.

Pre-requisites

A familiarity with maximum likelihood in a parametric setting, and basic properties of convex functions would be helpful.

Literature

1. Dümbgen, L., Samworth, R. and Schuhmacher, D. (2011) Approximation by log-concave distributions with applications to regression, *Ann. Statist.*, **39**, 702–730.

2. Groeneboom, P. and Jongbloed, G. (2014) *Nonparametric Inference under shape constraints.* Cambridge University Press.

3. Kim, A. K. H. and Samworth, R. J. (2016) Global rates of convergence in log-concave density estimation, *Ann. Statist.*, to appear.

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

B. Allanach

Quantum Field Theory is the language in which modern particle physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics. How these fields interact with a classical electromagnetic field is described.

Next, we introduce the path integral which is an alternative way of describing quantum fields. The path integral is fundamental in introducing interaction into quantum field theory. Interactions are described using perturbative theory and Feynman diagrams. This is first illustrated for theories with a purely scalar field interaction, and then for a couplings between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

Finally, the idea of loops in Feynman diagrams are explored and the question of the consequent infinities looked at. Ways of dealing with the infinities will be explored in the Advanced Quantum Field Theory course which follows on directly from this one.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Literature

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
2. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2010)
3. M. Srednicki, *Quantum Field Theory*, Cambridge University Press, (2007). (a free prelminary version is available here `http://web.physics.ucsb.edu/~mark/ms-qft-DRAFT.pdf`
4. M. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2014).

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Symmetries, Fields and Particles (M24)

N. Dorey

This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of $SU(2)$, the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of $SU(2)$ and introduce the tensor product of representations. The next part of the course develops the the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of $SU(3)$. The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate $SU(3)$ global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

Pre-requisites

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

Literature

1. J. Fuchs and C. Schweigert, *Lie Algebras and Representations.* Cambridge University Press, 2003.
2. H.F. Jones, *Representations and Physics.* 2nd edition. Taylor and Francis, 1998.
3. H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, 1999.

Additional support

A set of course notes will provided as handouts in the lectures. Printed notes of previous version of the course are also available on the Part III Examples and Lecture Notes webpage. Four examples sheets will be provided and four associated examples classes in moderate-sized groups will be given by graduate students.

Statistical Field Theory (M16)

M B Wingate

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory, and the construction of effective field theories. Statistical systems such as spin models provide the context for much of the course, but the connection to quantum field theory (in Euclidean spacetime) is made manifest using Landau & Ginzburg's scalar field theory.

The phenomenology of phase transitions is reviewed, leading to the introduction of the theory of critical phenomena. Landau theory is presented and applied to the Ising model. The classification of phase transitions and their relationship with critical points is presented. The renormalization group is introduced first in the context of the soluble 1D Ising model and then in general. The renormalization group is used for calculating properties of systems near a phase transition, for example in the Ising and Gaussian models, and the concepts of critical exponents, anomalous dimensions, and scaling are discussed.

Pre-requisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Clarendon Press (1992).
2. M Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1991).
3. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).
4. M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).
5. D Amit and V Martín-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).
6. L D Landau and E M Lifshitz, Statistical Physics, Pergamon Press (1996).

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Advanced Quantum Field Theory (L24)

DB Skinner

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalisation of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantising a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson Loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study Renormalization. Wilson's picture of Renormalisation is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the Renormalisation Group (RG) flow. The course explores renormalisation systematically, from the use of dimensional regularisation in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom", this phenomenon revolutionised our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrise possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Preliminary Reading

1. Zee, A., *Quantum Field Theory in a Nutshell*, 2$^{\text{nd}}$ edition, PUP (2010).

Literature

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).
2. Schwarz, M., *Quantum Field Theory and the Standard Model*, CUP (2014).
3. Srednicki, M., *Quantum Field Theory*, CUP (2007).
4. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).
5. Deligne, P. *et al.*, *Quantum Fields and Strings*, vol. 1, AMS (1999).

Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent Term. There will also be a general revision class during Easter Term.

Standard Model (L24)

C.E. Thomas

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the more general Advanced QFT course.

We begin by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, and spin-one gauge bosons). The parity P, charge-conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force and so it violates parity symmetry. We show how CP violation becomes possible when there are three generations of particles and describe its consequences.

Ideas of spontaneous symmetry breaking are applied to discuss the Higgs Mechanism and why the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. Recent measurements of what appear to be Higgs boson decays will be presented.

We show how to obtain cross sections and decay rates from the matrix element squared of a process. These can be computed for various scattering and decay processes in the electroweak sector using perturbation theory because the couplings are small. We touch upon the topic of neutrino masses and oscillations, an important window to physics beyond the Standard Model.

The strong interaction is described by quantum chromodynamics (QCD), the non-abelian gauge theory of the (unbroken) $SU(3)$ gauge symmetry. At low energies quarks are confined and form bound states called hadrons. The coupling constant decreases as the energy scale increases, to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Time permitting, we will discuss nonperturbative approaches to QCD. For example, the framework of effective field theories can be used to make progress in the limits of very small and very large quark masses.

Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advantageous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1995).
2. H. Georgi, *Weak Interactions and Modern Particle Theory*, Benjamin/Cummings (1984).
3. T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press (1984).

4. I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics*, IoP Publishing (2012) (two volumes or earlier 1989 edition in one volume).

5. F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, John Wiley and Sons (1984).

6. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press (1992).

Additional support

Four example sheets will be provided and four associated examples classes will be given as well as a revision class.

String Theory (L24)

M. Perry

String Theory supposes that elementary particles are excitations of a string, which could be open (with two endpoints) or closed. Closed strings have a massless spin-2 particle in their spectrum, which suggests that String Theory is a theory of quantum gravity. Open strings yield analogous generalisations of gauge theory, so a theory of open and closed strings is potentially one that can unify gravity with the forces of the standard model of particle physics. This course will introduce the geometry and dynamics of classical string and then move on to the quantisation of bosonic strings.

Various methods of quantisation of the NG string, including light-cone gauge, and "old covariant" and the possibly BRST method. This will reveal that there is a critical space-time dimension (26) and that the ground state is a tachyon. A study of the possible boundary conditions on open strings will suggest an interpretation in terms of branes. Superstring theory will be introduced, in the RNS formalism. The light-cone gauge will be used to show that the the critical dimension is 10. It will be explained briefly why superstring theories are tachyon-free and why there are five of them.

We will investigate the Virasoro-Shapiro amplitude for the scattering of closed-string tachyons of the bosonic string, and a discuss of some general features of string perturbation theory. This will include a look at the one-loop quantum corrections and why there are no UV divergences. Other topics that may be discussed are T-duality and how the five superstring theories are unified by M-Theory.

Pre-requisites

This course assumes you know the basics of (i) Special Relativity and (ii) Quantum Mechanics.

Literature

1. M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory: Vol. 1 and Vol. 2* CUP 1987.

2. R. Blumenhagen, D. Lüst and S. Theisen, *Basic Concepts of String Theory*, Springer 1989.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will also be a weekly office hour during the Lent term for questions about the lectures

Supersymmetry (E16)

F. Quevedo

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced. If time allows a short discussion of supersymmetry in higher dimensions will be briefly discussed.

Three examples sheets and examples classes will complement the course.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them.

Preliminary Reading

(a) The first chapters of `http://arxiv.org/abs/hep-ph/0505105`

Literature

For more advanced topics later in the course, it will helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as I know) Wess and Bagger.

(a) Course lecture notes from last year:
`http://www.damtp.cam.ac.uk/user/examples/3P7.pdf`

(b) Videos of a very similar lecture course: follow the links from
`http://users.hepforge.org/~allanach/teaching.html`

(c) Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics. An erratum can be found at
`http://www.phys.susx.ac.uk/~mpfg9/susyerta.htm`

(d) Introduction to supersymmetry, J.D. Lykken, `hep-th/9612114`. This introduction is good for extended supersymmetry and more formal aspects.

(e) Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.

(f) A supersymmetry primer, S.P. Martin, `hep-ph/9709256` is good and detailed for phenomenological aspects, although with the opposite sign metric to the course.

Classical and Quantum Solitons (E16)

N. S. Manton

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, they give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, and on Skyrmions in three dimensions. Solitons in gauge theories will also be mentioned.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Literature

(a) N. Manton and P. Sutcliffe, *Topological Solitons.* C.U.P., 2004 (Chapters 1,3,4,5,9).

(b) R. Rajaraman, *Solitons and Instantons.* North-Holland, 1987.

(c) A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects.* C.U.P., 1994 (Chapter 3).

Additional support

Two examples sheets will be provided and two associated examples classes will be given.

Relativity and Gravitation

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. A non-examinable course explores the application of spinor techniques in General Relativity.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/undergrad/course`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Essential*:	Classical Dynamics.
	Very helpful:	General Relativity, Stat. Phys., Electrodynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Cosmology (M24)

James Fergusson, David Marsh

This course covers the last 13.8 billion years of the evolution of your universe, from the initial inflationary quantum perturbations to the creation of galaxies we observe today. The course will follow the following format

(a) Geometry and Dynamics
(b) Inflation
(c) Cosmological Perturbation Theory
(d) Structure Formation
(e) Thermal History
(f) Initial Conditions from Inflation

Pre-requisites

This course is taught in a self contained manner so could be attempted by any sufficiently keen part III student but some basic knowledge of Relativity, Quantum Mechanics and Statistical Mechanics will likely be quite helpful.

Literature

(a) Dodelson, *Modern Cosmology*

(b) Kolb and Turner, *The Early Universe*

(c) Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

General Relativity (M24)

M. Dunajski and H. Reall

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

Pre-requisites

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before, then it is strongly recommended that you study an introductory book (e.g. Hartle or Schutz) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian Gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \to R^n$, and the Euler-Lagrange equations will be assumed.

Preliminary Reading

(a) J. B. Hartle, *An introduction to Einstein's General Relativity.* Addison-Wesley, 2003.

(b) B. Schutz, *A First Course in General Relativity.* Cambridge University Press, 2009.

Literature

(a) R. M. Wald, *General Relativity.* University of Chicago Press, 1984.

(b) J. M. Stewart, *Advanced General Relativity.* Cambridge University Press, 1993.

(c) C.W. Misner, K.S. Thorne and J. A. Wheeler, *Gravitation.* Freeman, 1973.

(d) S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Addison-Wesley, 2004.

This course is closest in content to Wald's book. Stewart's book gives a concise overview of the differential geometry in the first part of this course. Misner, Thorne and Wheeler's book is particularly useful for the sections on gravitational radiation and the Newtonian limit. Carroll's book is a very readable introduction.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Course website: `http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html`

Black Holes (L24)

J. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for the recent advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

The following topics will be discussed:

(a) Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
(b) The initial value problem, strong cosmic censorship.
(c) Causal structure, null geodesic congruences, Penrose singularity theorem.
(d) Penrose diagrams, asymptotic flatness, weak cosmic censorship.
(e) Reissner-Nordström and Kerr black holes.
(f) Energy, angular momentum and charge in curved spacetime.
(g) Positivity of energy theorem.
(h) The laws of black hole mechanics. The analogy with laws of thermodynamics.
(i) Quantum field theory in curved spacetime. The Hawking effect and its implications.

Pre-requisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

(a) R.M. Wald, *General relativity*, University of Chicago Press, 1984.
(b) S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
(c) V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
(d) N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
(e) R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these. A revision class will be held in the Easter term.

Spinor Techniques in General Relativity (L24)

Non-Examinable (Graduate Level)

Irena Borzym (12 Lectures) and Peter O'Donnell (12 Lectures)

Spinor structures and techniques are an essential part of modern mathematical physics. This course provides a gentle introduction to spinor methods which are illustrated with reference to a simple 2-spinor formalism in four dimensions. Apart from their role in the description of fermions, spinors also often provide useful geometric insights and consequent algebraic simplifications of some calculations which are cumbersome in terms of spacetime tensors.

The first half of the course will include an introduction to spinors illustrated by 2-spinors. Topics covered will include the conformal group on Minkowski space and a discussion of conformal compactifications, geometry of scri, other simple simple geometric applications of spinor techniques, zero rest mass field equations, Petrov classification, the Plucker embedding and a comparison with Euclidean spacetime. More specific references will be provided during the course and there will be worked examples and handouts provided during the lectures.

The second half of the course will include: Newman-Penrose (NP) spin coefficient formalism, NP field equations, NP quantities under Lorentz transformations, Geroch-Held-Penrose (GHP) formalism, modified GHP formalism, Goldberg-Sachs theorem, Lanczos potential theory, Introduction to twistors. There will be no problem sets.

Pre-requisites

The Part 3 general relativity course is a prerequisite.

No prior knowledge of spinors will be assumed.

Literature

Introductory material.

(a) L. P. Hughston and K. P. Tod, *Introduction to General Relativity.* Freeman, 1990.

(b) C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation.* Freeman, 1973.

Best Course Reference Text for Lectures 1 to 12.

J.M. Stewart, *Advanced General Relativity.* CUP, 1993.

Best Course Reference Text for Lectures 13 to 24.

P O'Donnell, *Introduction to 2-spinors in general relativity.* World Scientific, 2003.

Reading to complement course material.

(a) Penrose and Rindler, *Spinors and Spacetime Volume 1.* Cambridge Monographs on Mathematical Physics, 1987.

(b) S. Ward and Raymond O. Wells, *Twistor Geometry and Field theory.* Cambridge Monographs on Mathematical Physics, 1991 .

(c) Robert J. Baston, Michael G. Eastwood, *The Penrose Transform.* Clarendon Press, 1989.

(d) S. A Huggett and P. Tod, *Introduction to Twistor Theory.* World Scientific, 2003.

(e) R.M. Wald, *General Relativity.* World Chicago UP, 1984.

(f) S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime.* CUP, 1973.

Advanced Cosmology (L24)

Anthony Challinor and Tobias Baldauf

This course will take forward at much greater depth some of the topics in modern cosmology covered in the Michaelmas Term course. The prediction from fundamental theory for the statistical properties of the primordial perturbations remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave background (CMB). This course will develop the mathematical tools and physical understanding necessary for research in this very active area.

Cosmic microwave background

- Statistics of random fields
- Relativistic kinetic theory
- The Boltzmann equation
- The CMB temperature power spectrum
- Photon scattering and diffusion
- Primordial gravitational waves and the CMB
- CMB Polarization

Inflationary theory and Large-Scale Structure

- Primordial non-Gaussianities
- Effective field theory of inflation
- CMB bispectrum and optimal estimators
- Modelling late time non-linearities in large-scale structure
- Effective field theory of large-scale structure
- Tracers of large-scale structure and the peak formalism

Pre-requisites

Material from the Michaelmas term *Cosmology* is essential. Familiarity with introductory Quantum Field Theory is recommended.

Literature

Textbooks

(a) Dodelson, S., *Modern Cosmology*, Elsevier (2003).
(b) Mukhanov, V., *Physical Foundation of Cosmology*, Cambridge (2005).
(c) Weinberg, S., *Cosmology*, Oxford University Press (2008).
(d) Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, Freeman (1973).
(e) Durrer, R., *The Cosmic Microwave Background*, Cambridge (2008).

Useful references

(a) Bardeen, J.M., *Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure*, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request.)
(b) Mukhanov, V.F., Feldman, H.A., and Brandenberger, R.H., *Theory of cosmological perturbations*, Physics Reports, 215, 203 (1992).

(c) Ma, C., and Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gauges*, Astrophysical Journal, 455, 7 (1995) [astro-ph/9506072].

(d) Hu, W. and White, M., *CMB anisotropies: Total angular momentum method*, Physical Review D, 56, 596 (1997) [astro-ph/9702170].

(e) Hu, W. and White, M., *A CMB polarization primer*, New Astronomy, 2, 323 (1997) [astro-ph/97006147].

(f) Maldacena, J., *Non-gaussian features of primordial fluctuations in single field inflationary models*, Journal of High Energy Physics, 5, 13 (2003).

(g) Chen, X., *Primordial Non-Gaussianities from Inflation Models* [arxiv:1002.1416].

(h) Wang, Yi., *Inflation, Cosmic Perturbations and Non-Gaussianities* , [arXiv:1303.1523] (Conference Lecture Notes).

(i) Senatore, L., Smith, K., Zaldarriaga, M. *Non-Gaussianities in Single Field Inflation and their Optimal Limits from the WMAP 5-year Data*, Journal of Cosmology and Astroparticle Physics, 1, 28 (2010) [arXiv:0905.3746]

(j) Ligouri, M., Sefusatti, E., Fergusson, J.R., and Shellard, E.P.S., *Primordial Non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure*, Advances in Astronomy, 2010, 73 (2010) [arxiv:1001.4707]

(k) Bernardeau, F., Colombi, S., Gaztanaga, E., Scoccimarro, R., *Large-scale structure of the Universe and cosmological perturbation theory*, Physics Reports, 367 (2002) [arXiv:astro-ph/0112551]

(l) Hertzberg, M., *Effective field theory of dark matter and structure formation: Semianalytical results*, Physical Review D, 89, 4 (2014) [arXiv:1208.0839]

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrophysics

Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Astrophysical Fluid Dynamics (M24)

Gordon Ogilvie

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

Provisional synopsis

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

Pre-requisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

(a) Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.

(b) Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.

(c) Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press. Available as an e-book from

`http://ebooks.cambridge.org`

(d) Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.

(e) Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.

(f) Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics.* J. Plasma Phys. **82**, 205820301.

Additional support

Four example sheets will be provided and four associated classes will be given by the lecturer. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of(well over 1000) extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course,including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

(a) Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation

(b) Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits

(c) Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift

(d) Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits

(e) Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide

(f) Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces

(g) Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments

(h) Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems

(i) Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

(a) Murray C. D. and Dermott S. F.,*Solar System Dynamics.* Cambridge University Press,1999.

(b) Armitage P. J., *Astrophysics of Planet Formation.* Cambridge University Press, 2010.

(c) de Pater I. and Lissauer J. J., *Planetary Sciences.* Cambridge University Press, 2010.

(d) Valtonen M. and Karttunen H., *The Three-Body Problem.* Cambridge University Press, 2006.

(e) Seager S., *Exoplanets.* University of Arizona Press, 2011.

(f) Perryman M., *The Exoplanet Handbook.* Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a mathematical description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These basic equations have to be supplemented by a number of appropriately chosen equations describing the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be shown; polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

The only way in which we may test stellar structure and evolution theory is through comparison of the theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Pre-requisites

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Preliminary Reading

(a) Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.

(b) Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

(a) Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.

(b) Iben, I. *Stellar Evolution Physics, Vol. 1 and 2.* Cambridge University Press, 2013.

(c) Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.

(d) Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

Convection and Magnetoconvection (L16)

Prof. M.R.E. Proctor

Convection is the name given to the means used by fluids to transfer heat when fluid flow is more effective than conduction. In a fluid layer, for example, between horizontal boundaries held at fixed temperatures, convection occurs when the temperature difference is sufficiently large. The onset of convection can be thought of as an instability of pattern forming type, and there are many interesting questions that can be asked: What are the pattern and horizontal scale of convection near onset? How does the heat transfer depend on the temperature difference? How do the simple patterns seen at onset break down? What is the effect on convection of other physical effects such as rotation, and what happens when there are two sources of buoyancy, such as thermohaline convection, or when there are other constraints present such as rotation and/or magnetic fields?

The course will address many of these issues. Though convection requires that fluid density depend on temperature and so be non-uniform, most of the course will use the Boussinesq approximation, in which the fluid may be treated as incompressible except for the buoyancy term. This approximation is a good one for laboratory liquids and gives a good guide to many aspect of convection for which the approximation is not accurate.

There will be three problem sheets and associated examples classes.

Desirable Previous Knowledge

Knowledge of fluid dynamics and dynamical systems would be an advantage.

Introductory Reading

(a) Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability. Dover

(b) Drazin, P and Read, W. Hydromagnetic stability (chapter 2). CUP

Reading to complement course material

(a) Weiss, N.O and Proctor, M.R.E, Magnetoconvection. CUP

(b) Getling, A.V. Rayleigh-Benard convection: structures and dynamics. World Scientific

(c) Hoyle, R. Pattern Formation. CUP

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Pre-requisites

The Michaelmas term course on Structure and Evolution of Stars is very useful but not absolutely essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Literature

(a) Pringle J. E. and Wade R. A., *Interacting Binary Stars.* CUP.

Reading to complement course material

(a) Eggleton P. P., *Evolutionary Processes in Binary and Multiple Stars.* CUP.

Additional support

Three examples sheets will be provided and three associated two-hour classes will be given. There will be a two-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Discs of matter in orbital motion around a massive central body occur in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies; they reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure, thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics in protoplanetary disks.
- Compressible dynamics: density waves, gravitational instability and 'gravitoturbulence' in planetary rings and protoplanetary discs.
- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, 'dead zones' in protoplanetary discs.

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the magnetorotational instability.

Literature

(a) Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.

(b) Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Optical and infrared astronomical telescopes and instruments (L16)

Ian Parry

Astronomy is an observational science. Our understanding of the universe beyond the Earth comes mostly from interpreting the electromagnetic radiation we see coming from the sky. This course is about the equipment and techniques that we use to collect and measure the optical and near infra-red component of this radiation (approximately 0.3 to 5 microns in wavelength).

The material presented will give the student a thorough understanding of how telescopes and their instruments actually work. An important aim of the course is to quantify how well they work leading to an understanding of what defines the state-of-the-art and what its limitations are.

Specific topics will be selected from the following list;

(a) Introduction: Effects of the Earths atmosphere, transparency, seeing, refraction, dispersion, basic definition of magnitudes.

(b) Positional astronomy and coordinate systems: Sidereal time, right ascension, declination, hour angle, aberration of starlight, spherical trigonometry, great circles, small circles, spherical triangles, cosine and sine rules, tangent plane.

(c) Optics: geometrical optics, lens-makers equation, principle planes, focal length, f-number, aberrations, paraxial approximation, ray-tracing, image planes, pupil planes, conjugate planes, simple lens design, achromatic lenses, the Petzval lens, methods of designing complex optical systems, tendue, physical optics, quantum optics, Fourier treatment of wave propagation.

(d) Telescopes. Refractors, reflectors, parabolic reflectors, Ritchey-Chretien telescopes, 3-mirror anastigmats, the Schmidt telescope, ground-based telescopes, telescope mounts, space-based telescopes.

(e) Detectors: photo-electrons, useful semiconductor types, readout architectures, image intensifiers, linearity, dynamic range, quantum efficiency, pixel-to-pixel variations, readout noise, dark-current, sensitivity to cosmic rays, defects, charge transfer, artefacts.

(f) Adaptive Optics: wavefront sensors, deformable mirrors, control algorithms, Kolmogorov turbulence, Zernike polynomials.

(g) Imagers: eyepieces, the eye, magnification, collimators, cameras, detector matching, image scale, field of view, filters, magnitude systems. Fabry Perot interferometers, polarimetry.

(h) Coronagraphs: Fourier modelling, speckles, occulters, lyot-stop, apodization,

(i) Spectrographs: Dispersive spectrometers (long-slit, multi-slit, multi-fibre, echelle, integral field), disperser types, grating equation, spectro-polarimetry, Fourier-transform spectrometers.

(j) Interferometers:, Stellar interferometry: Fizeau-Stephan interferometer, Michelson stellar interferometer, closure-phase, non-redundant masks.

(k) Signal-to-noise ratio: exposure time, Poisson noise, systematic errors, beam-switching, cryogenics.

(l) Future projects: E-ELT, LSST, JWST, HDST.

Pre-requisites

This course is self-contained.

Literature

(a) Roy, A.E., & Clarke, D., Astronomy Principles and Practice, 4th ed., Institute of Physics, 2003.

(b) Kitchin, C.R.: Astrophysical Techniques, 4th ed., Institute of Physics, 2003.

(c) Smart, W.M., Spherical Astronomy, 6th ed., Cambridge University Press, 1977.

(d) Saha S. K., Diffraction-limited imaging with large and moderate telescopes, World Scientific, New Jersey, 2007.

(e) Born & Wolf, Principles of Optics, 7th ed., Cambridge University Press, 2002

(f) Optics : E.Hecht

(g) Speckle Phenomenon in Optics : J.W.Goodman

(h) Astronomical Optics : D.J.Shroeder

(i) Astronomical Techniques : W.A.Hiltner

(j) Optical Detectors for Astronomy : J.W.Beletic & P.DAmico

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (L24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths, and rocky exoplanets), and (4) relating atmospheres and interiors to planet formation. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres of all kinds. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self contained and basic concepts will be introduced for completeness.

Literature

(a) Chapters on exoplanetary atmospheres and interiors in the book Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Most of these chapters are available publicly on the astro-ph arXiv.

(b) Seager, S., Exoplanet Atmospheres: Physical Processes, Princeton Series in Astrophysics (2010).

(c) Exoplanets, University of Arizona Press (2011), ed. S. Seager.

(d) de Pater, I. and Lissauer J., Planetary Sciences, Cambridge University Press (2010).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Galactic Astronomy and Dynamics (L24)

Wyn Evans

Astrophysics provides many examples of complex dynamical systems. This course covers the mathematical tools to describe Galaxies as well as reviewing their observational properties. The behaviour of these systems is controlled by Newton's laws of motion and Newton's law of gravity. Galaxies are dynamically very young, a typical star like the Sun having orbited only thirty or so times around the galaxy. The motions of stars in Galaxies are described using classical statistical mechanics, since the number of stars is so great. The study of large assemblies of stars interacting via long-range forces provides many unusual examples of cooperative phenomena, such as bars and spiral structure. The interplay between astrophysical dynamics and modern cosmology is also important – much of the evidence for dark matter is dynamical in origin.

(a) Observational overview. Stellar populations in galaxies, galaxy morphology and classification. Dust and gas in galaxies. Scaling Laws.
(b) Theory of the gravitational potential. Poisson's equation. Spherical, spheroidal and disk-like systems.
(c) Regular and chaotic orbits, the epicyclic approximation, surfaces of section, integrals of motion, action-angle coordinates, adiabatic invariance.
(d) Collisionless stellar dynamics, the Boltzmann equation, the Jeans Theorem, the Jeans equations, equilibrium models, astrophysical applications.
(e) Collisional dynamics, the Fokker-Planck equation, dynamical friction.
(f) Globular cluster evolution, evaporation and ejection, the gravothermal catastrophe, the effect of hard and soft binaries.
(g) Galactic stability, the Jeans length, theories of spiral structure, the role of resonances.
(h) The Milky Way Galaxy, the Local Group. Disk, bar, bulge and halo of the Milky Way

Pre-requisites

This course is suitable for applied mathematicians and astrophysicists. Although the course is self-contained, familiarity with Lagrangian & Hamiltonian mechanics and mathematical methods would be useful.

Preliminary Reading

(a) Harwit M., 1982 *Cosmic Discovery: The Search, Scope and Heritage of Astronomy*, Basic Books
(b) Elmegreen D.M., 1997 *Galaxies and Galactic Structure*, Prentice Hall
(c) Sparke L., Gallagher J., 2007 *Galaxies in the Universe*, Cambridge University Press

Literature

(a) Bertin G., 2000, *The Dynamics of Galaxies*, Cambridge University Press
(b) Binney J., Tremaine S., 2007, *Galactic Dynamics*, Princeton University Press
(c) Heggie D., Hut P. 2003, *The Million Body Problem*, Cambridge University Press
(d) Murray C, Dermott S., 1999, *Solar System Dynamics*, Cambridge University Press

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will also be a two-hour revision class in the Easter Term.

Quantum Computation, Information and Foundations

Quantum Computation (M16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. Notable examples include integer factorisation (and consequent efficient breaking of commonly used public key crypto systems) and database searching. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

The course will cover the following topics:

Notion of qubits, quantum logic gates, circuit model of quantum computation. Basic notions of quantum computational complexity, oracles, query complexity.

The quantum Fourier transform. Exposition of fundamental quantum algorithms including the Deutsch-Jozsa algorithm, Shors factoring algorithm, Grovers searching algorithm.

A selection from the following further topics (and possibly others):
(i) Quantum teleportation and the measurement-based model of quantum computation;
(ii) Lower bounds on quantum query complexity;
(iii) Phase estimation and applications in quantum algorithms;
(iv) Quantum simulation for local hamiltonians.

Pre-requisites

It is desirable to have familiarity with the basic formalism of quantum mechanics especially in the simple context of finite dimensional state spaces (state vectors, Dirac notation, composite systems, unitary matrices, Born rule for quantum measurements). Prerequisite notes will be provided on the course webpage giving an account of the necessary material including exercises on the use of notations and relevant calculational techniques of linear algebra. It would be desirable for you to look through this material at (or slightly before) the start of the course. Any encounter with basic ideas of classical theoretical computer science (complexity theory) would be helpful but is not essential.

Literature

(a) Nielsen, M. and Chuang, I., *Quantum Computation and Quantum Information.* CUP, 2000.

(b) Kaye, P., Laflamme, R. and Mosca, M. *An Introduction to Quantum Computing.* OUP, 2007.

(c) John Preskill *Lecture Notes on Quantum Information Theory*, available at
`http://www.theory.caltech.edu/people/preskill/ph219/`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Space-time in Light of Particle Physics (M8)

Non-Examinable (Part III Level)

J. Brian Pitts

Pre-requisites

Imagine a world in which gravitation theory and particle physics have always mixed freely. These lectures will sketch such a world by taking a particle physics-flavoured look at gravitational theory, including General Relativity and some (perhaps) serious competition. Wigner's taxonomy in terms of mass and spin provides a starting place. How do we know that gravity is a tensor and not a scalar or vector? How (if at all) do we know that gravity is massless? What can we say about gravity and space-time if gravity is not presumed to be exceptional? Could one plausibly arrive at Einstein's equations? Do Einstein's principles, if not assumed, reappear as theorems? What does one make of conservation laws and Noether's theorems? How do spinors fit in, especially given nonlinear group realizations? We will also glance at Einstein's process of discovery, including his rediscovered 'physical strategy' involving an analogy to electromagnetism and attention to conservation laws. The mathematics will be classical field theory, which is surprisingly rich. An Essay will be associated with this course.

Literature

(a) Preskill-Thorne foreword to the *Feynman Lectures on Gravitation*, 1995.

(b) D. Giulini, 'What Is (Not) Wrong with Scalar Gravity?', *Studies in History and Philosophy of Modern Physics*, 39:154-180, 2008, gr-qc/0611100v2.

(c) P. G. O. Freund and Y. Nambu, 'Scalar Fields Coupled to the Trace of the Energy-Momentum Tensor', *Physical Review*, 174:1741–1743, 1968.

(d) Markus Fierz and Wolfgang Pauli, 'On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field', *Proceedings of the Royal Society (London) A*, 173:211-232, 1939.

(e) Robert H. Kraichnan, 'Special Relativistic Derivation of Generally Covariant Gravitation Theory.' *Physical Review*, 98:1118–1122, 1955.

(f) V. I. Ogievetsky and I. V. Polubarinov, 'Interacting Field of Spin 2 and the Einstein Equations', *Annals of Physics*, 35:167–208, 1965.

(g) P. G. O. Freund, A. Maheshwari, and E. Schonberg, 'Finite-Range Gravitation', *Astrophysical Journal*, 157:857–867, 1969.

(h) David G. Boulware and Stanley Deser, 'Can Gravitation Have a Finite Range?', *Physical Review D*, 6:3368–3382, 1972.

(i) J. Renn and T. Sauer, 'Heuristics and Mathematical Representation in Einstein's Search for a Gravitational Field Equation', in H. Goenner, J. Renn, J. Ritter and T. Sauer, *The Expanding Worlds of General Relativity*, pp. 87-125, Einstein Studies, volume 7, Birkhäuser, Boston, 1999.

(j) J. Brian Pitts, 'Einstein's Physical Strategy, Energy Conservation, Symmetries, and Stability: "but Grossmann & I believed that the conservation laws were not satisfied"', *Studies in History and Philosophy of Modern Physics*, 54:52-72, 2016.

Philosophical Aspects of Quantum Field Theory (L8)

Non-Examinable (Part III Level)

J. Butterfield

Quantum field theory has for many decades been the framework for several basic and outstandingly successful physical theories. Nowadays, it is being addressed by philosophy of physics (which has traditionally concentrated on conceptual questions raised by non-relativistic quantum mechanics and relativity). This course will introduce this literature. More specifically, we will address the following topics: particle vs. field, including second vs. field quantization; localisation; and the algebraic approach to quantum field theory.

Pre-requisites

There are no formal prerequisites. Previous familiarity with the quantum field theory, such as provided by the Part III courses, will be helpful.

Preliminary Reading

This list of introductory reading is approximately in order of increasing difficulty.

(a) S. Weinberg (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027; and in Cao ed.

(b) D. Wallace (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Available online at: http://arxiv.org/pdf/quant-ph/0112148v1

(c) R. Clifton and H. Halvorson (2001), 'Are Rindler quanta real? Inequivalent particle concepts in quantum field theory', *British Journal for Philosophy of Science*, **52**, pp 417-470. Sections 1, 2.1, 2.2, 3.1, 3.2. Available online at: http://arxiv.org/abs/quant-ph/0008030

Literature

This list of readings to complement course material is approximately in order of increasing difficulty.

(a) D. Wallace (2001), 'Emergence of particles from bosonic quantum field theory'. Available online at: http://arxiv.org/abs/quant-ph/0112149

(b) T. Cao, (ed.) *The Conceptual Foundations of Quantum Field Theory.* Cambridge University Press, 1999.

(c) L. Ruetsche, *Interpreting Quantum Theories.* Oxford University Press, 2011.

(d) W. Greiner. *Relativistic Quantum Mechanics.* 2nd edition. Springer 1997.

(e) W. Greiner and J. Reinhardt. *Field Quantization.* Springer 1996.

(f) R. Haag. *Local Quantum Physics: fields, particles, algebras.* Springer 1992.

(g) A. Duncan, *The Conceptual Framework of Quantum Field Theory.* Oxford University Press, 2012.

Additional support

A Part III essay will be offered in conjunction with this course.

Applied and Computational Analysis

Inverse Problems in Imaging (M24)

Martin Benning, Matthias Ehrhardt

Solving an inverse problem is the task of computing an unknown physical quantity that is related to given, indirect measurements via a forward model. Inverse problems appear in a vast majority of applications, including imaging (Computed Tomography (CT), Positron Emission Tomography (PET), Magnetic Resonance Imaging (MRI), Electron Tomography (ET), microscopic imaging, geophysical imaging), signal- and image-processing, computer vision, machine learning and (big) data analysis in general, and many more.

Inverting a forward model however is not straightforward in most relevant applications, for two basic reasons: either a (unique) inverse model simply does not exist, or existing inverse models heavily amplify small measurement errors. In this course we are going to address the mathematical aspects of inverse problems, and discuss the concept of regularisation for finding stable approximations of the inverse of a specific forward model.

Pre-requisites

This course assumes basic knowledge in analysis and linear algebra, as well as their numerical counterparts. In addition to that, basic programming skills in MATLAB are required.

Additional knowledge in partial differential equations, functional analysis, variational calculus, image processing or (convex) optimisation is beneficial, but not mandatory.

Literature

(a) H. W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems.* Vol. 375, Springer Science & Business Media, 1996, ISBN: 9780792341574.

(b) P. C. Hansen. *Discrete Inverse Problems: Insight and Algorithms.* Fundamentals of Algorithms, SIAM Philadelphia, 2010, ISBN: 9780898718836.

(c) O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen. *Variational Methods in Imaging.* Applied Mathematical Sciences, Springer New York, 2008, ISBN: 9780387309316.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Distribution Theory and Applications (L16)

A.C.L. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the use of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will look at the Sobolev spaces $H^s(\mathbf{R}^n)$ and $H^s_{\mathrm{loc}}(X)$ and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. In the final part of the course we will study Hörmander's oscillatory integrals.

Pre-requisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Analysis/Methods). No knowledge of functional analysis is assumed.

Preliminary Reading

(a) Friedlander & Joshi, *Introduction to the Theory of Distributions.* Cambridge University Press, 1998.

(b) Lighthill, *Introduction to Fourier Analysis and Generalised Functions.* Cambridge University Press, 1958.

(c) Folland, *Introduction to Partial Differential Equations.* Princeton University Press, 1995.

Literature

(a) Hörmander, *The Analysis of Partial Differential Operators: Vol I.* Springer Verlag, 1985.

(b) Reed & Simon, *Methods of Modern Mathematical Physics: Vol I-II.* Academic Press, 1979.

(c) Tréves, *Linear Partial Differential Equations with Constant Coefficients.* Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available.

Topics in Convex Optimisation (L16)

Hamza Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover some techniques known as *convex relaxations*, to deal with optimisation problems involving polynomials, which are in general intractable. The emphasis of the course will be on semidefinite programming which is a far-reaching generalization of linear programming. A tentative list of topics that we will cover include:

- From linear programming to conic programming. Duality theory.
- Semidefinite optimisation and convex relaxations. Sums-of-squares and moment problems.
- Applications: binary quadratic optimisation and rounding methods (e.g., Goemans-Williamson rounding), stability of dynamical systems, matrix completion/low-rank matrix recovery, etc.

Pre-requisites

This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Literature

(a) A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, 2001 (`http://dx.doi.org/10.1137/1.9780898718829`).
(b) G. Blekherman, P. Parrilo, R. Thomas, *Semidefinite optimization and convex algebraic geometry*, SIAM 2013 (`http://dx.doi.org/10.1137/1.9781611972290`).
(c) S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004 (`http://web.stanford.edu/~boyd/cvxbook/`).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Boundary Value Problems for Linear PDEs (L16)

Iasonas Hitzazis

Recent developments in the area of the so-called *integrable nonlinear* Partial Differential Equations (PDEs) have led to the emergence of a new method for solving boundary value problems, which is usually referred to as the *Unified Transform* (UT).

The UT will be implemented to:

(a) Linear evolution PDEs in one spatial variable formulated either on the half-line or on a finite interval. Examples include the heat equation and the Stokes equation (linearised version of the KdV).

(b) Linear elliptic PDEs in two spatial variables formulated in the interior of a convex polygon. Examples include the Laplace, the modified Helmholtz, and the Helmholtz equations.

For the above problems, in addition to presenting integral representations of the solution, simple numerical techniques for the effective computation of the solution will also be introduced.

Pre-requisites

The course only requires some elementary knowledge of complex analysis.

Literature

(a) A.S. Fokas, *A unified approach to boundary value problems.* 1st edition. SIAM, 2008.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Topics in Mathematics of Information (L24)

C. Poon

In an increasingly data driven world, there is an essential need for efficient acquisition techniques, the correct representation of signals and the ability to extract meaningful information from signals. This is an introductory course to the mathematics behind such techniques. This course will cover the following topics:

- The representation and approximation of signals, in particular, linear and nonlinear approximation in Fourier and wavelet bases.
- Efficient data acquisition by exploiting the inherent sparse structure of signals, in particular, an introduction to compressed sensing.
- Topics in inverse problems, in particular, the use of total variation regularization to exploit the geometric structures of the underlying signals.

Pre-requisites

This course assumes basic knowledge in analysis and linear algebra. Additional knowledge in partial differential equations, functional analysis, variational calculus is beneficial, but not mandatory.

Literature

(a) Foucart, Simon and Holger Rauhut. *A mathematical introduction to compressive sensing.* Birkhuser, 2013.

(b) Mallat, Stéphane. *A wavelet tour of signal processing: the sparse way.* Academic press, 2008

(c) Chambolle, Antonin, et al. *An introduction to total variation for image analysis.* Theoretical foundations and numerical methods for sparse recovery 9.263-340 (2010): 227.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Numerical Solution of Differential Equations (L24)

Arieh Iserles

The course will address modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, with an emphasis on broad mathematical principles underlying their construction and analysis.

Pre-requisites

Although prior knowledge of *some* numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

Literature

(a) U. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, 2008.

(b) A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (2nd edition), Cambridge University Press, 2006.

Additional support

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be weekly examples' classes, starting from the third week, as well as a revision supervision in the Easter Term.

Compressed Sensing and Sampling Theory (L16)

Non-Examinable (Graduate Level)

Anders Hansen

This is a graduate course on sampling theory and compressed sensing for use in signal processing and medical imaging. Compressed sensing is a theory of randomisation, sparsity and non-linear optimisation techniques that breaks traditional barriers in sampling theory. Since its introduction in 2004 the field has exploded and is rapidly growing and changing. Thus, we will take the word contemporary quite literally and emphasise the latest developments, however, no previous knowledge of the field is assumed. Although the main focus will be on compressed sensing, it will be presented in the general framework of sampling theory. The course will also present related areas of sampling theory such as generalised sampling.

The course will be fairly self contained, and applications will be emphasised (in particular, signal processing, Magnetic Resonance Imaging (MRI) and X-ray Tomography). The lectures will cover the most up to date research, and although this is a Part III course, it is also aimed at Phd students and post docs who are interested in using compressed sensing and generalised sampling in their research. Students from other disciplines than mathematics are encouraged to participate.

Pre-requisites

Sampling theory and compressed sensing require a mix of mathematical tools from approximation theory, harmonic analysis, linear algebra, functional analysis, optimisation and probability theory. The course will contain discussions of both finite-dimensional and infinite-dimensional/analog signal models and thus linear algebra, Fourier analysis and functional analysis (at least basic Hilbert space theory) are important. The course will be self-contained, but students are encouraged to refresh their memories on properties of the Fourier transform as well as basic Hilbert space theory. Some basic knowledge of wavelets is useful as well as basic probability.

Preliminary Reading

For a quick and dense review of basic Fourier analysis and functional analysis chapters 5 and 8 of ”Real Analysis” (Folland) are good choices. For an introductory exposition to Hilbert space theory one may use ”An Introduction to Hilbert Space” (Young). And for a review of wavelets see chapters 1 and 2 of ”A First Course on Wavelets” (Hernandez, Weiss). The course will cover some of the chapters of ”Compressed Sensing” (Eldar, Kutyniok), so to get a feeling about the topic one may consult chapter 1 as a start.

(a) Eldar, Y and Kutyniok, G., Compressed Sensing, CUP

(b) Folland, G. B., Real Analysis, Wiley.

(c) Hernandez, E. and Weiss, G., A First Course on Wavelets, CRC

(d) Young, N., An Introduction to Hilbert Space, CUP

Literature

The following reading list complements the course material.

(a) Adcock, B and Hansen, A., Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon, Appl. Comp. Harm. Anal., 32 (2012)

(b) Candès, E. and Romberg, J. and Tao, T., Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory 52 (2006)

(c) Donoho, D., Compressed sensing, IEEE Trans. Inform. Theory 52 (2006)

(d) Körner, T. W., Fourier Analysis, CUP

(e) Reed, M. and Simon, B., Functional Analysis, Elsevier

Additional support

As this is a non-examinable course there will be no examples classes, however, there will be several computer tutorials where practical implementations and real world examples will be discussed. There will also occasionally be lectures given by people from other groups outside of mathematics using compressed sensing in practice.

Continuum Mechanics

The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Hydrodynamic Stability (M24)

Colm-cille Caulfield

Developing an understanding by which "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After an introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently "high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry. They typically demonstrate the key role played by the redistribution of vorticity within the flow, and such vortical flow instabilities often trigger the complex, yet hugely important process of "transition to turbulence".

A hierarchy of mathematical approaches will be discussed to address a range of "stability" problems, from more traditional concepts of "linear" infinitesimal normal mode perturbation energy growth on laminar parallel shear flows to transient, inherently nonlinear perturbation growth of general measures of perturbation magnitude over finite time horizons where flow geometry and/or fluid properties play a dominant role. The course will also discuss in detail physical interpretations of the various flow instabilities considered, as well as the industrial and environmental application of the results of the presented mathematical analyses.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

(a) F. Charru *Hydrodynamic Instabilities* CUP 2011.

(b) P. G. Drazin & W. H. Reid *Hydrodynamic Stability* 2nd edition. CUP 2004.

(c) P. Huerre *Open Shear Flow Instabilities* in *Perspectives in Fluid Dynamics: A Collective Introduction to Current Research* ed. G. K. Batchelor, H. K. Moffatt & M. G. Worster CUP 2000.

(d) P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows.* Springer 2001.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading

(a) D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7

(b) G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.

(c) L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Literature

(a) J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).

(b) S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)

(c) C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).

(d) O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Biological Physics and Complex Fluids (M24)

Raymond Goldstein, Eric Lauga

This course will provide an overview of the physical and mathematical description of both living and synthetic small-scale complex systems. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology. The first half of the course will give an overview of the fundamental physical process at play in biology. After an introduction to statistical mechanics, the topics will include molecular interactions, polymers, elasticity, chemical dynamics, and dynamics. The second part of the course will build on the first half and bridge the gap from the microscopic physics to the continuum scale in order to describe in detail the flow of complex, non-Newtonian fluids and the theory of phoretic motion relevant to colloidal science.

Pre-requisites

Undergraduate fluid dynamics, vector calculus and mathematical methods. Some familiarity with statistical physics will be helpful.

Literature

(a) P. Nelson. Biological Physics. W.H. Freeman (2007).

(b) J.D. Murray. Mathematical Biology I. and II. Springer (2007, 2008).

(c) National Committee for Fluid Mechanics Films on "Rheological Behavior of Fluids" and "Low Reynolds Number Flow" at: `http://web.mit.edu/hml/ncfmf.html`

(d) F. A. Morrison. Understanding Rheology, OUP (2001).

(e) B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts and P. Walter. Molecular Biology of the Cell. 5th edition. Garland Science (2007).

(f) J.N. Israelachvili. Intermolecular and Surface Forces. 2nd edition. Academic Press (1992).

(g) E.J.W. Verwey and J.Th.G. Overbeek. Theory of the Stability of Lyophobic Colloids. Elsevier (1948).

(h) M. Doi and S.F. Edwards. The Theory of Polymer Dynamics. OUP (1986).

(i) R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd ed, Wiley (1987).

(j) D. Boal. Mechanics of The Cell, CUP (2001) .

(k) R. G. Larson. The Structure and Rheology of Complex Fluids, OUP (1999).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Perturbation Methods (M16)

L.J. Ayton & S.J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. [6]
- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, and Domb-Sykes plots. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKBJLG' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [4]

Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

(a) Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*

(b) Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991). *This is the book of the course; some view it as somewhat terse.*

(c) Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975). *This is the original book on perturbation methods; somewhat dated, but still a useful read.*

Reading to Complement Course Material

(a) Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

(b) Boyd, J.P., *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, **56**, 1-98 (1999). Also available at
`http://hdl.handle.net/2027.42/41670` and
`http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`

(c) Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Fluid dynamics of the solid Earth (L24)

Jerome A. Neufeld & M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field. At very much longer time scales, radiogenic heating of the solid mantle drives solid-state convection resulting in plume-like features possibly responsible for features such as the Hawaiian sea mounts. Nearer the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record. Upwelling mantle material also drives partial melting of mantle rocks resulting in compaction, and ultimately in the propagation of viscous melt through the elastic crust. On the Earth's surface, and at very much faster rates, the same physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

Pre-requisites

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Literature

(a) M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)

(b) H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)

(c) D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP (2002)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Quantum Fluids (L16)

Prof. Natalia Berloff

Superfluidity is the central topic across many fields of physics, including condensed matter, quantum field theory, critical phenomena, classical hydrodynamics and nuclear matter. In the last two decades the field has undergone an important transformation combining theory with experimental realisations and potential applications. The course presents an overview of superfluidity with emphasis on properties of various quantum fluids from superfluid helium to atomic condensates and solid state condensates. In particular, the following topics will be covered in some detail:

- **Theory of classical complex-valued matter fields and their topological properties.** Neutral matter field. Classical Hamiltonian Formalism. Normal Modes by Bogoliubov Transformation. Nonrelativistic Limit for a Complex-Valued Scalar Field. Gross-Pitaevskii Equation. Vortices. Landau criterion. Matter Field under rotation. Superfluidity at Finite Temperatures and Hydrodynamics.
- **Fundamental macroscopic properties of superfluids.** Basics of Superfluid Hydrodynamics, Thermomechanical Effects, Hydrodynamic Hamiltonian and Action. Superfluid Phase Transition. Berezinskii-Kosterlitz-Thouless Phase Transition. Multicomponent systems.
- **Quantum-mechanical aspects and macrodynamic of quantum fluids.** Dynamics of Vortices. Vortex-Phonon Interaction. Turbulence, Kelvin-Wave Cascade.
- **Theory of the weakly interacting Bose gas.** Kinetics of Bose?Einstein Condensation. Weak turbulence and cascades. The Anderson mechanism. Superfluid States in Optical-Lattice Emulators.
- **Solid-state condensates.** Superfluid States in Nature and the Laboratory. Stable, Metastable, and Unstable Elementary Excitations. Exciton-polariton condensates. Quantum simulators.

Pre-requisites

Basic knowledge of quantum mechanics, statistical physics and fluid dynamics

Literature

(a) B. Svistunov E. Babaev, N. Prokof'ev, *"Superfluid States of Matter"* CRC Press, Taylor and Fransis Group, 2015

(b) L.M. Pismen *"Vortices in nonlinear fields: from liquid crystals to superfluids; from non-equilibrium patterns to cosmic strings"*, International series of monographs in physics 100, Clarendon Press Oxford, 1999.

(c) A. Leggett, *"Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems,"* Oxford Graduate Texts 2006.

(d) R.J. Donnelly, *"Quantized Vortices in Helium II"* Cambridge University Press, Cambridge, 1991.

(e) P.H. Roberts and N.G. Berloff, *"Nonlinear Schrodinger equation as a model of superfluid helium,"* in "Quantized Vortex Dynamics and Superfluid Turbulence" edited by C.F. Barenghi, R.J. Donnelly and W.F. Vinen, Lecture Notes in Physics, volume 571, Springer-Verlag, 2001.
`http://www.damtp.cam.ac.uk/user/ngb23/publications/review.pdf`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Theoretical Physics of Soft Condensed Matter (L16)

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, molten polymers and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, with noise, whose equations of motion are stochastic PDEs. In cases where the system eventually reaches a steady state of thermal equilibrium, the form of these PDEs is strongly constrained by the requirement that the Boltzmann distribution is regained in that limit. Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

Important models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field. Further steps, towards renormalization group theory and the detailed exploration of critical phenomena, will be noted but not actually taken in this course.

I also plan to address the continuum theory of nematic liquid crystals, which spontaneously break rotational but not translational symmetry. This will be developed at mean field level, allowing a discussion of topological defects in soft matter and their associated mathematical structure such as homotopy classes.

If time permits, I will describe extensions of the same general approach to systems whose microscopic dynamics does not have time-reversal symmetry, such as self-propelled colloidal swimmers. These systems do not have a Boltzmann distribution in steady state; without that constraint, new field theories arise that are the subject of ongoing research.

Caveat: This is a new course: the planned content may be altered as it proceeds.

Pre-requisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the following Michaelmas Term courses although none are prerequisites: Statistical Field Theory; Biological Physics and Complex Fluids; Slow Viscous Flow; Quantum Field Theory.

Preliminary Reading

(a) D. Tong *Lectures on Statistical Physics*
`http://www.damtp.cam.ac.uk/user/tong/statphys.html`

Before embarking on this course you do need to understand the equation $F = -k_BT \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Familiarity with the Landau theory of phase transitions is highly desirable. We will not need much abstract thermodynamics (e.g. Maxwell relations) but you do need to know the zeroth, first and second laws. These lecture notes are an excellent resource for revising and reviewing the key material.

(b) M. E. Cates *Complex Fluids: The Physics of Emulsions.*
`http://arxiv.org/abs/1209.2290`

This set of lecture notes addresses only one part of the course (emulsions); it goes into more depth in that area than we will, but with more words and significantly less mathematics. It takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics when needed. Despite all this, these notes would make useful preliminary reading and gives an idea of the types of problem we will address.

Literature

(a) I am not aware of any books that treat this material at the right level. I will issue an update at the start of the course if I find one. Meanwhile, the only suggestion is....

(b) P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, topological defects all feature in the description of condensed matter systems at $\hbar = 0$. More for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be at least one one-hour revision class in the Easter Term.

Environmental Fluid Dynamics (L16)

A.W. Woods

Understanding and predicting the impact of human activity on the environment is a critical challenge in our time. This course introduces some of the basic fluid dynamics associated with man's environment. The course will include a discussion of turbulent buoyant plumes and gravity currents, contrasting these flows with flows which arise from more distributed sources of buoyancy. We will also introduce the ideas of turbulent dispersion and turbulent shear driven mixing.

These processes form the building blocks for a discussion of urban fluid mechanics, which will form an important part of the course. We will discuss flows in buildings, in which there is a competition between forced ventilation flows and buoyancy driven flows arising from sources of heating or cooling. We will also discuss the flow and ventilation of tunnels, and the fluid dynamics of fire and smoke plumes, and the relevance of this modelling approach to interpret the dispersal of air-borne microbes, with applications in health care. We will then explore models for mixing in the outside urban environment, in which the presence of buildings and other 'porous' obstacles leads to substantial mixing and dispersion of the air and suspended pollutants by wind driven flows.

We will then turn to a series of other specific fluid dynamical processes in man's environment, including the dynamics of single and two phase plumes and jets, with relevance for bubble-driven mixing of water reservoirs, mixing in chemical systems, and also for modelling the dynamics of rain clouds, volcanic ash clouds, oil-field blow-outs and hydrothermal plumes. We will also explore some of the controls on the propagation of dense gas releases such as occured during the Lake Nyos CO2 explosion and we will discuss the relevance for volcanic ash flows and submarine turbidite flows, in which the presence of particles control the advance of the flow.

Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

(a) J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press, 1979.

Additional support

In addition to the lectures, three examples sheets will be provided and three associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Convection and Magnetoconvection (L16)

Prof. M.R.E. Proctor

Convection is the name given to the means used by fluids to transfer heat when fluid flow is more effective than conduction. In a fluid layer, for example, between horizontal boundaries held at fixed temperatures, convection occurs when the temperature difference is sufficiently large. The onset of convection can be thought of as an instability of pattern forming type, and there are many interesting questions that can be asked: What are the pattern and horizontal scale of convection near onset? How does the heat transfer depend on the temperature difference? How do the simple patterns seen at onset break down? What is the effect on convection of other physical effects such as rotation, and what happens when there are two sources of buoyancy, such as thermohaline convection, or when there are other constraints present such as rotation and/or magnetic fields?

The course will address many of these issues. Though convection requires that fluid density depend on temperature and so be non-uniform, most of the course will use the Boussinesq approximation, in which the fluid may be treated as incompressible except for the buoyancy term. This approximation is a good one for laboratory liquids and gives a good guide to many aspect of convection for which the approximation is not accurate.

There will be three problem sheets and associated examples classes.

Desirable Previous Knowledge

Knowledge of fluid dynamics and dynamical systems would be an advantage.

Introductory Reading

(a) Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability. Dover

(b) Drazin, P and Read, W. Hydromagnetic stability (chapter 2). CUP

Reading to complement course material

(a) Weiss, N.O and Proctor, M.R.E, Magnetoconvection. CUP

(b) Getling, A.V. Rayleigh-Benard convection: structures and dynamics. World Scientific

(c) Hoyle, R. Pattern Formation. CUP

Demonstrations in Fluid Mechanics. (L8)

Non-Examinable (Part III Level)

Prof. S.B. Dalziel, Dr. J.A. Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- porous media and carbon sequestration;
- fluid flow and elastic deformation;
- ventilation and industrial flows;
- rotationally dominated flows;

- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Pre-requisites

Undergraduate Fluid Dynamics.

Literature

(a) M. Van Dyke. An Album of Fluid Motion. Parabolic Press.

(b) G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.

(c) M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.