UNIVERSITY OF CAMBRIDGE

M. Phil. in STATISTICAL SCIENCE

Monday, 8 June, 2009   9:00 am to 11:00 am

OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

> **You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

1   Baron Kiwi, the shoe-polish billionaire, appoints an investment manager to handle his investments for him. His initial wealth is w_0, and the manager can invest in a riskless asset returning a constant interest rate r, and in d risky assets, whose prices $S_t \equiv (S_t^1, \ldots, S_t^d)^T$ evolve as log-Brownian motions:

$$dS_t^i = S_t^i \left(\sum_{j=1}^d \sigma_{ij} dW_t^j + \mu_i dt \right) \qquad (i = 1, \ldots, d),$$

where the W^j are independent standard Brownian motions. The wealth process generated by the manager satisfies

$$dw_t = r w_t dt + \sum_i w_t \pi_t^i \left(\sum_j \sigma_{ij} dW_t^j + (\mu_i - r) dt \right),$$

where π_t^i denotes the fraction of wealth invested in asset i at time t. Baron Kiwi wants the investment manager to increase his wealth by time T, when his daughter is due to buy a University. If the manager's (admissible) investment results in a wealth w_T at time T, then the Baron has promised to pay the manager

$$y_T \equiv a w_T \exp\left(-\tfrac{1}{2} \varepsilon \int_0^T |\sigma^T \pi_s|^2 \, ds \right),$$

where $a > 0$, $\varepsilon > 0$, and the Baron's intention in reducing the payment by the exponential factor is to make the manager less keen to follow risky strategies.

If the manager's objective is to maximise $EU(y_T)$, where

$$U(x) = \frac{x^{1-R}}{1-R}$$

for some positive $R \neq 1$, prove that his optimal policy is exactly that which would be followed by an agent with utility

$$U_0(x) = \frac{x^{1-R-\varepsilon}}{1-R-\varepsilon}$$

and objective to maximize $EU_0(w_T)$.

Determine this optimal policy explicitly.

2   An agent may invest in a single risky asset whose price process S evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt)$$

and in a riskless bank account. The interest paid on the bank account is at rate $r + \varepsilon > r$ until a random time τ which has an exponential distribution with mean λ^{-1}, and after that time the interest rate falls to r and stays there. The agent's objective is to maximise

$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where c_t is the rate of consumption withdrawal, $\rho > 0$ is constant, and $U(x) = x^{1-R}/(1-R)$ for some $R > 0$ different from 1.

(i) Consider first the case where the interest rate has already changed. Write down the agent's value function and optimal investment/consumption policy. [*You may use without derivation any standard results from the course.*]

(ii) Now consider the case where the interest rate has not yet changed. Obtain an equation which characterises the agent's value function and optimal investment/consumption policy, and find the value function as explicitly as you can.

3   (i) Suppose that an agent may invest in a riskless bank account yielding interest at constant rate r, and in a stock, whose price S_t at time t evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt).$$

If the agent aims to maximise the objective

(*)
$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

where $U'(x) = x^{-R}$ for some positive $R \neq 1$, and c_t is rate of consumption withdrawal, write down the dynamics for the agent's wealth, and by finding the Hamilton-Jacobi-Bellman equations for this problem, or otherwise, show that the agent's optimal policy is to invest a fixed proportion π_M of his wealth in the risky asset, and to consume at a rate $c_t = \gamma_M w_t$ for some constant γ_M, where you should identify π_M and γ_M explicitly in terms of the parameters of the problem.

[*You should assume that $\rho + (R-1)(r + (\mu - r)^2/2\sigma^2 R) > 0$. You are not required to give a verification proof.*]

(ii) Suppose that there is a single productive asset in the world, which generates consumption goods at rate δ_t, where for some constants $\sigma > 0$, α,

$$d\delta_t = \delta_t(\sigma dW_t + \alpha dt).$$

There is a market in shares in the productive asset, and in riskless borrowing/lending of the consumption good.

Consider a representative-agent economy, where the single agent has objective (*). Show that if the share price process $(S_t)_{t \geqslant 0}$ is

$$S_t = \frac{\delta_t}{\rho + (R-1)(\alpha - \sigma^2 R/2)}$$

and the riskless rate of interest is constant, equal to

$$r^* = \rho + \alpha R - \tfrac{1}{2}(1 + R)\sigma^2 R$$

then the markets clear; the representative agent faced with these prices will at all times hold one share in the productive asset, and will put no wealth into riskless borrowing/lending. Show also that the optimal consumption process is δ_t.

[*You should assume that $\rho + (R-1)(\alpha - \sigma^2 R/2) > 0$.*]

4   (i) Suppose that $X_t = W_t + \alpha t$, where W is a standard Brownian motion, and let $\mathcal{X}_t \equiv \sigma(X_u : 0 \leqslant u \leqslant t)$. The drift α is constant but unknown; it takes one of the values a or $-a$, where $a > 0$. Each of the two possible values has equal prior probability. Mr Bayes observes the process X, and makes inference on the unknown drift α. Show that after observing until time t his posterior for α takes the form

$$P[\, \alpha = a \mid \mathcal{X}_t \,] = \frac{e^{aX_t}}{e^{aX_t} + e^{-aX_t}}.$$

Deduce that the dynamics of X in the filtration $(\mathcal{X}_t)$ can be expressed as

$$dX_t = d\hat{W}_t + a \tanh aX_t \, dt,$$

where $\hat{W}$ is a $(\mathcal{X}_t)$-Brownian motion.

(ii) Mr Bayes wishes to invest in a market where there is a riskless bank account yielding interest at constant rate r, and a single risky asset whose price S_t at time t is given as

$$S_t = S_0 \exp(\sigma X_t + (\mu - \tfrac{1}{2}\sigma^2)t),$$

where X is as in part (i). His objective is to maximize $E \int_0^\infty e^{-\rho t} U(c_t) \, dt$, where $\rho > 0$ is constant, c_t is his rate of consumption withdrawal, and $U'(x) = x^{-R}$ for some positive $R \neq 1$.

   Derive the Hamilton-Jacobi-Bellman equation for Mr Bayes' value function, and obtain an expression for the optimal investment in the risky asset.

   Assume now that $\mu > r$, and $0 < R < 1$. Another agent, Mr Smart, knows that in fact $\alpha = a$; his objective is the same as Mr Bayes'. By considering Mr Smart's value function V, and the process

$$Y_t = \int_0^t e^{-\rho t} U(c_t) \, dt + e^{-\rho t} V(w_t),$$

where c is Mr Bayes' optimal consumption process, and w is Mr Bayes' optimal wealth process, show that Y is a supermartingale.

   Deduce that Mr Bayes' value cannot exceed Mr Smart's value.

END OF PAPER

Optimal Investment