M.PHIL. IN STATISTICAL SCIENCE

Tuesday 3 June 2008 9.00 to 12.00

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt no more than **FOUR** *questions.*

There are **SIX** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let M be a continuous local martingale.

(a) Prove that if M is of finite variation then M must be indistinguishable from 0.

(b) Show that there is at most one nondecreasing adapted process A (up to indistinguishability) such that $M^2 - A$ is a continuous local martingale. (However, do not attempt to prove the existence of such a process).

(c) Show that M is a true martingale bounded in L^2 if and only if $\mathbb{E}([M]_\infty) < \infty$. All limits must be properly justified.

2 Let $d \geq 3$ and let B be a Brownian motion in $\mathbb{R}^d$, started at $B_0 = \bar{x}$ where $\bar{x} = (x, 0, \ldots, 0) \in \mathbb{R}^d$ for some $x > 0$. Let $\| \cdot \|$ be the Euclidean norm on $\mathbb{R}^d$. Let $\tau_a = \inf\{t > 0 : \|B_t\| = a\}$, and for a stopping time T, define B^T to be the process $(B_{t \wedge T}, t \geq 0)$.

(a) Let $D = \mathbb{R}^d \setminus \{0\}$ and let $h : D \to \mathbb{R}$ be defined by $h(x) = \|x\|^{2-d}$. Show that h is harmonic on D and that $M_t = \|B_t^{\tau_a}\|^{2-d}$ is a local martingale for all $a \geq 0$. Is M a true martingale? (Consider the cases $a > 0$ and $a = 0$ separately).

(b) Use (a) to show that for any $a < b$ such that $0 < a < x < b$,

$$\mathbb{P}_{\bar{x}}(\tau_a < \tau_b) = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

where ϕ is the function defined on $\mathbb{R}_+$ by $\phi(a) = a^{2-d}$. Conclude that if $x > a > 0$

$$\mathbb{P}_{\bar{x}}(\tau_a < \infty) = (a/x)^{d-2}.$$

3

(a) Let $(Z_t, t \geq 0)$ be a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale such that $Z = \mathcal{E}(M)$, where $\mathcal{E}(M)$ denotes the exponential local martingale associated with M:

$$\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}[M]_t)$$

and $[M]$ denotes the quadratic variation process of M. [*Hint: for the existence part, define* $M_t = \ln Z_0 + \int_0^t \frac{dZ_s}{Z_s}$*, and then apply Itô's formula to* $\ln Z_t$*.*]

(b) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ such that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}$. Show that if $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ for all $0 \leq t \leq T$, then Z is a nonnegative $\mathbb{P}$-martingale.

(c) Assuming that Z is strictly positive almost surely and continuous, what can we say about the relations between semi-martingales with respect to $\mathbb{P}$ and $\mathbb{Q}$? [You may assume any standard result of the course.]

4 Let $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ be two independent one-dimensional standard Brownian motions.

(a) Assume that $X_0 = Y_0 = 0$ almost surely. Show that if $\tau_x = \inf\{t > 0 : X_t = x\}$ for some $x > 0$ then τ_x has the same distribution as $x^2/\mathcal{N}^2$ where $\mathcal{N}$ is a standard normal random variable. Use this result and a scaling argument to conclude that

$$Y_{\tau_x} \stackrel{d}{=} x\frac{\mathcal{N}}{\mathcal{N}'}$$

where $\stackrel{d}{=}$ means that the distributions of the left and right hand side are equal, and where $\mathcal{N}, \mathcal{N}'$ are independent standard normal random variables. The distribution of $\mathcal{N}/\mathcal{N}'$ is known as the Cauchy distribution.

(b) Regard $Z_t = (X_t, Y_t)$ as a two-dimensional Brownian motion, and assume that $X_0 = -x$ and $Y_0 = 0$, with $x > 0$. Let $\tau = \inf\{t > 0 : \Re Z_t \geq 0\}$, where $\Re z$ denote the real part of $z \in \mathbb{C}$. Deduce from part (a) that

$$Y_\tau \stackrel{d}{=} xC$$

where C has the Cauchy distribution.

(c) Let $x > 0$ and let $\varepsilon = e^{-x}$. Let Z'_t be a two dimensional Brownian motion with $Z'_0 = (\varepsilon, 0)$. Let θ_ε denote the number of windings of Z' prior to time $T = \inf\{t > 0 : \|Z'_t\| = 1\}$. That is, each time Z' completes a winding around zero in the counterclockwise direction, this adds $+1$ to θ_ε, and each clockwise winding about 0 adds -1 to θ_ε. Use (b) and conformal invariance of Brownian motion to show that as $\varepsilon \to 0$,

$$\frac{\theta_\varepsilon}{\log \varepsilon} \xrightarrow{d} \frac{1}{2\pi} C$$

where C is a Cauchy random variable, and where $\xrightarrow{d}$ stands for convergence in distribution. [*Hint: consider* $\exp(X_t + iY_t)$ *where* $Z_t = (X_t, Y_t)$ *is the Brownian motion from part (b)*]. This result is Spitzer's law on windings of Brownian motion.

5 Consider the stochastic differential equation (SDE):

$$\begin{cases} dZ_t = \sqrt{Z_t} dB_t \\ Z_0 = z > 0 \end{cases} \tag{1}$$

where B is a Brownian motion.

(a) Define what the terms "strong solutions" and "pathwise uniqueness" mean. Explain why the theorem in the course guaranteeing pathwise uniqueness and existence of strong solutions does not apply to the SDE (1).

(b) Let Z be a solution to (1) started from $Z_0 = z$ driven by a Brownian motion B, and let Z' be a solution started from $Z'_0 = z'$ driven by a Brownian motion B'. Show that if B and B' are independent, then $Z + Z'$ is a weak solution to (1) started from $z + z'$, driven by some Brownian motion β to be determined.

(c) Let $\varepsilon > 0$ and let $\sigma_\varepsilon(x)$ be a function such that σ_ε is a continuously differentiable function with $\sigma_\varepsilon(x) = \sqrt{x}$ for all $x \geq \varepsilon$ and $\sigma_\varepsilon(x) = 0$ for all $x \leq \varepsilon/2$, while $\sigma_\varepsilon(x) \geq 0$ for all $x \geq 0$. (We assume that such a function exists). Show that the modified SDE:

$$\begin{cases} dZ_t^\varepsilon = \sigma_\varepsilon(Z_t^\varepsilon) dB_t \\ Z_0^\varepsilon = z > 0 \end{cases} \tag{2}$$

has pathwise unique, strong solutions. Fix $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and B an $(\mathcal{F}_t)$-Brownian motion and let Z^ε denote a strong solution to (2). (Note that we use the same driving Brownian motion B for varying ε). Let $T_\varepsilon = \inf\{t > 0 : Z_t^\varepsilon \leq \varepsilon\}$. Show that if $\varepsilon' < \varepsilon$, then Z^ε and $Z^{\varepsilon'}$ must agree on $[0, T_\varepsilon]$. Use this to construct a strong solution to (1).

6 Let H be a real-valued adapted continuous process, which is uniformly bounded, and let $(B_t, t \geq 0)$ be a one-dimensional standard Brownian motion.

(a) Using Cauchy-Schwarz's inequality and Jensen's inequality, or otherwise, show that

$$\mathbb{E}\left(\left| \frac{1}{B_\varepsilon} \int_0^\varepsilon (H_u - H_0) dB_u \right|^{1/4} \right) \to 0$$

as $\varepsilon \to 0$.

(b) Use (a) to prove that for all $t > 0$,

$$\lim_{\varepsilon \to 0} \frac{1}{B_{t+\varepsilon} - B_t} \int_t^{t+\varepsilon} H_s dB_s = H_t$$

in probability.

END OF PAPER