UNIVERSITY OF CAMBRIDGE

M. Phil. in STATISTICAL SCIENCE

Friday 10 June, 2005   9 to 11

MATHEMATICAL MODELS IN FINANCIAL MANAGEMENT

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the conditions under which discrete time stochastic process $\mathbf{M} := \{\mathbf{M}_n : n = 0, \ldots, N\}$ with filtration $\{\mathcal{F}_n\}$ is a $\mathbb{P}$-*martingale*.

Show that the process $\mathbf{V} := \{\mathbf{V}_n : n = 0, \ldots, N\}$ with $\mathbf{V}_n := \sum_{i=1}^{n} \boldsymbol{\phi}_i(\mathbf{M}_i - \mathbf{M}_{i-1})$ is a discrete time $\mathbb{P}$-martingale if $\mathbf{M}$ is a $\mathbb{P}$-martingale and $\boldsymbol{\phi}$ is a bounded previsible process (i.e. $\boldsymbol{\phi}$ is $\mathcal{F}_{i-1}$ measurable denoted $\boldsymbol{\phi} \in \mathcal{F}_{i-1}$).

Using an approximation by such sums of martingale differences, give an heuristic argument to show that for a continuous time stochastic Ito integral

$$\mathbb{E} \int_0^t g(s, \mathbf{W}_s) d\mathbf{W}_s = 0$$

and

$$\mathbb{E} \left(\int_0^t g(s, \mathbf{W}_s) d\mathbf{W}_s \right)^2 = \mathbb{E} \int_0^t g(s, \mathbf{W}_s)^2 ds$$

for a suitable function g with $\int_0^t \mathbb{E} g(s, \mathbf{W}_s)^2 ds < \infty$.

The *Ornstein-Uhlenbeck* (OU) $\mathbf{X}$ process satisfies the SDE

$$d\mathbf{X}_t = \kappa(\theta - \mathbf{X}_t)dt + \sigma d\mathbf{W}_t, \quad \mathbf{X}_0 = x \quad a.s.,$$

where κ, θ and σ are constants and $\mathbf{W}$ is a Wiener process. Solve this SDE by setting $\mathbf{Y}_t := e^{\kappa t}\mathbf{X}_t$ and using Ito's lemma to show that

$$\mathbf{X}_t = \theta + (x - \theta)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} d\mathbf{W}_s.$$

Find the mean and variance of $\mathbf{X}_t$.

Let $\mathbf{C}$ be a *Cox-Ingersoll-Ross* (CIR) process satisfying the SDE

$$d\mathbf{C}_t = a(b - \mathbf{C}_t)dt + \sigma\sqrt{\mathbf{C}_t} d\mathbf{W}_t.$$

By considering $\mathbf{Z}_t := \sqrt{\mathbf{C}_t}$ derive an SDE for $\mathbf{Z}$. Choosing the parameters so that $2ab = \sigma^2$, show that $\mathbf{Z}$ is an OU process and hence find $\mathbb{E}(\mathbf{C}_t)$.

2 The *Black-Scholes-Merton* PDE for the value of a contract $V(S,t)$ depending on an asset price S at time t for an asset paying continuous dividends at a constant rate q in an environment with constant interest rate r is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0,$$

where σ is a constant volatility of S.

Show that if V can be separated as a product of the form

$$V(S,t) = f(S)g(t),$$

then g, f and their derivatives $\dot{g}, f', f''$ must be related by

$$-\frac{\dot{g}(t)}{g(t)} + r = \frac{\frac{1}{2}\sigma^2 S^2 f''(S) + (r-q)Sf'(S)}{f(S)}.$$

Explain why g and f must therefore satisfy the *ordinary* differential equations

$$\begin{aligned} \dot{g}(t) &= (r-\lambda)g(t) \\ \frac{1}{2}\sigma^2 S^2 f''(S) + (r-q)Sf'(S) - \lambda f(S) &= 0 \end{aligned}$$

for some constant λ.

Using these ordinary differential equations, find *all* solutions of the Black-Scholes-Merton equation under the assumption $r > 0$ for each of the following cases:

(a) V is independent of S

(b) V is independent of t

(c) $f(S) = kS$, where k is a constant

(d) $f(S) = kS^\alpha$, where α is a non-zero real constant.

A *power call* price $C_\beta(S,t)$ is defined for $\beta > 0$ to be a solution of the Black-Scholes-Merton equation with payoff at $t = T$ given by

$$C_\beta(S,T) := \max[S^\beta - K^\beta, 0].$$

Similarly the *power put* price $P_\beta(S,t)$ has payoff

$$P_\beta(S,T) := \max[K^\beta - S^\beta, 0].$$

By simplifying the value of $C_\beta(S,t) - P_\beta(S,t)$ deduce the *put-call parity* relationship for such power instruments.

3 Consider the pricing of a European call option consistently with a given volatility smile or skew. To this end assume a Black- Scholes-type price satisfying

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad 0 < S, \quad 0 < t < T$$

with a suitable terminal condition $V(S,T) = (S-K)^+$ and a given *local volatility* function σ satisfying $\sigma_- \leqslant \sigma(S,t) \leqslant \sigma_+$ (e.g. from a previous calibration step), where $\sigma_\pm$ are known constants. Suppose also $r > 0$.

(a) It is convenient to introduce the transformations $x := \ln S$, $u(x,t) := e^{r(T-t)}V(e^x,t)$. Write down the PDE for u and the corresponding SDE for $\mathbf{x}$.

(b) Consider a numerical scheme in which the new variables are restricted to a finite interval (which is large enough not to introduce any significant errors by setting asymptotic boundary values) with an equidistant space grid with n points (mesh width Δx) and m time steps Δt. Let u_i^j be the approximation at x_i and t_j, $i = 1,\ldots,n$, $j = 1,\ldots,m$ and write down the discretisation using central differences for the space derivatives and *explicit* Euler time stepping.

(c) An initial approximate solution is performed on a very coarse grid with given values of Δx and Δt. To improve the accuracy, Δx is repeatedly divided by 2 while monitoring the numerical solution evaluated at the money, which is assumed to coincide with a grid point. Does the solution converge? If so, towards what?

(d) Repeating the same procedure by refining Δt instead, does the solution converge? If so, towards what?

(e) Finally, the space step is divided by 2 and the time step by 4 during refinements. Comparing the difference between the solutions of two subsequent refinement levels, which (convergence) behaviour is observed?

(f) In order to hedge the contract, the previous numerical solution is used to approximate the Δ (as before) by a central difference. What is the order of convergence for Δ? Explain. Can reliable information about Γ be retrieved from this numerical solution? Why (not)?

(g) Noting that explicit finite differences can be viewed as a *trinomial tree*, write the recursive scheme in the form

$$u_i^{j-1} = p_{i,d}^j u_{i-1}^j + p_{i,m}^j u_i^j + p_{i,u}^j u_{i+1}^j$$

and sketch the corresponding tree, giving the weights p_i^j explicitly in terms of the parameters. Under which conditions (on Δx and Δt) can the weights be seen as probabilities? How do these relate to the stability of the finite difference scheme?

4

(a) In the single factor *Heath-Jarrow-Morton* (HJM) model, given an initial forward rate curve $f(0,\cdot)$, the *forward rate* for each maturity T evolves as

$$d\mathbf{f}(t,T) = \boldsymbol{\alpha}(t,T)dt + \boldsymbol{\sigma}(t,T)d\mathbf{W}_t \quad 0 \leqslant t \leqslant T,$$

where $\mathbf{W}$ is a Wiener process and $\boldsymbol{\alpha}_t$ and $\boldsymbol{\sigma}_t$ can depend on $\mathbf{W}$ and $\mathbf{f}(\cdot,T), T \geqslant t$, up to time t. Give expressions for the *bond* price $\mathbf{P}(t,T)$, the *short rate* $\mathbf{r}_t := \mathbf{f}(t,t)$ and, defining the *cash bond* price $\mathbf{B}_t := \exp\left(\int_0^t \mathbf{r}_s ds\right)$, the *discounted bond* price $\mathbf{Z}_t(t,T) := \mathbf{B}_t^{-1}\mathbf{P}(t,T)$.

(b) Defining $\boldsymbol{\Sigma}(t,T) := -\int_t^T \boldsymbol{\sigma}(t,u)du$, under the risk neutral measure Q, $\mathbf{P}$ has dynamics

$$d\mathbf{P}(t,T) = \mathbf{P}(t,T)[\mathbf{r}_t dt + \boldsymbol{\Sigma}(t,T)d\mathbf{W}_t].$$

If $\mathbf{X}$ is the single payoff of a derivative maturing at $T > t$ show that its current price at t is given by

$$\mathbf{V}(t,\mathbf{r}) = \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T \mathbf{r}_s ds}\mathbf{X}|\mathcal{F}_t\right].$$

(c) Give an expression for the dynamics of the short rate $\mathbf{r}$ under Q. Is it necessarily Markov? Explain your answer.

5

(a) Describe briefly the concept of credit rating by the major ratings agencies for the issuers of corporate and sovereign bonds.

In the modelling of corporate default events in a single ratings class a simple model is to assume that defaults occur according to a Poisson process of constant rate λ. Let $\mathbf{N}_t$ denote the number of default events in time t. Then the probability distribution of $\mathbf{N}_t$ is

$$P(\mathbf{N}_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n \geqslant 0.$$

(b) By considering the probability that there are no events by time t, show that the time for the first event $\boldsymbol{\tau}_1$ has an exponential distribution with parameter λ and density

$$f_{\tau_1}(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

Find the mean and variance of $\boldsymbol{\tau}_1$.

(c) By considering the sequence $\{\boldsymbol{\tau}_i\}$ of independent exponential random variables, where $\boldsymbol{\tau}_i$ is the time between the i-th and $i-1$-th event, find the maximum likelihood estimator $\boldsymbol{\lambda}$ for λ. Explain how alternatively you could use the Poisson process sampled at unit time intervals to estimate λ. Is there any reason for preferring one technique over the other?

(d) State the central limit theorem for a sum of independent and identically distributed random variables and hence show that $\mathbf{T} := \sum_{i=1}^{n} \boldsymbol{\tau}_i$ is approximately normally distributed. Use this result to find an approximate 95% confidence interval for the parameter λ in the exponential distribution.

(e) Consider the relevance of the above to the annual *value at risk* of a commercial loan book for a single ratings class when losses given default are drawn independently (of each other and of default times measured in years) from the distribution of $\mathbf{X}$ with mean μ and variance σ^2.

Note that if Φ denotes the cumulative distribution function for the standard normal then

$$\Phi(1.282) = 0.9, \Phi(1.645) = 0.95, \Phi(1.96) = 0.975, \Phi(2.326) = 0.99.$$

END OF PAPER