UNIVERSITY OF CAMBRIDGE

## M. Phil. in STATISTICAL SCIENCE

Wednesday 2 June, 2004 1.30 to 3.30

# LARGE DEVIATIONS AND QUEUES

*Attempt* **THREE** *questions.*

*There are* **four** *questions in total.*

*The questions carry equal weight.*

*While rigorous answers are preferred, heuristic answers will still gain partial credit.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $(X_n,\ n \in \mathbb{N})$ satisfy a large deviations principle in some space $\mathcal{X}$ with good rate function $I$. Let $f$ be a bounded continuous function $\mathcal{X} \to \mathbb{R}$.

(a) Let $C_1, \ldots, C_m$ be closed subsets of $\mathcal{X}$ with $\bigcup_i C_i = \mathcal{X}$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \leqslant \max_{1\leqslant i\leqslant m} \Big\{ \sup_{x\in C_i} f(x) - \inf_{x\in C_i} I(x) \Big\}.$$

(b) Let $f(\mathcal{X})$ be contained in the interval $[a, b]$. Pick any $\varepsilon > 0$ and define the closed intervals

$$D_i = [a + (i-1)\varepsilon, a + i\varepsilon], \quad i = 1, \ldots, \lceil (b-a)/\varepsilon \rceil.$$

Let $C_i = f^{-1}(D_i)$. Using your answer to part (a), or otherwise, prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \leqslant \sup_{x\in\mathcal{X}} \Big[ f(x) - I(x) + \varepsilon \Big].$$

(c) Pick any $\hat{x} \in \mathcal{X}$ and any $\varepsilon > 0$. Define the open interval

$$D = (f(\hat{x}) - \varepsilon, f(\hat{x}) + \varepsilon).$$

Let $B = f^{-1}(D)$. Using this set, or otherwise, prove that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \geqslant f(\hat{x}) - I(\hat{x}) - \varepsilon.$$

(d) Deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} = \sup_{x\in\mathcal{X}} \Big[ f(x) - I(x) \Big].$$

**2** A sequence of random variables $(X_n,\ n \in \mathbb{N})$ taking values in a metric space $\mathcal{X}$ is said to have *Hurstiness* $H \in (0,1)$ if the following three conditions are satisfied:

- $(X_n,\ n \in \mathbb{N})$ satisfies a large deviations principle with good rate function $I$ at speed $n^{2(1-H)}$;
- there is some $\hat{x} \in \mathcal{X}$ such that $0 < I(\hat{x}) < \infty$;
- there is some $\mu \in \mathcal{X}$ such that $I(x) = 0$ only if $x = \mu$.

Suppose $(X_n,\ n \in \mathbb{N})$ has Hurstiness $H$. Let $G > H$, $G \in (0,1)$, and define the good rate function.

$$I'(x) = \begin{cases} 0 & \text{if } I(x) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

(a) Prove that for any closed set $C$

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in C) \leqslant - \inf_{x\in C} I'(x).$$

(b) Using your answer to (a), or otherwise, show that if $D$ is an open set containing $\mu$ then

$$\mathbb{P}(X_n \notin D) \to 0.$$

Hence (or otherwise) show that for any open set $E$

$$\liminf_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in E) \geqslant - \inf_{x\in E} I'(x).$$

(c) Suppose that $(X_n,\ n \in \mathbb{N})$ has Hurstiness $H$, that $(Y_n,\ n \in \mathbb{N})$ has Hurstiness $G$, that $X_n$ is independent of $Y_n$, and that both take values in $\mathbb{R}$. Show that $(X_n + Y_n,\ n \in \mathbb{N})$ has Hurstiness equal to the greater of $H$ and $G$.

*Note. You should mention any general results you use, but you need not state them formally. Recall that $(X_n,\ n \in \mathbb{N})$ is said to satisfy an LDP with rate function $I$ and speed $n^{2(1-H)}$ if for all measurable sets $B \subset \mathcal{X}$*

$$\begin{aligned} - \inf_{x\in B^\circ} I(x) &\leqslant \liminf_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in B) \\ &\leqslant \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in B) \leqslant - \inf_{x\in \bar{B}} I(x)\,. \end{aligned}$$

**3** Packets arrive at an Internet router as a Poisson process of rate $\lambda$ packets per second. Each packet has a payload; payload sizes are independent of each other and of the arrival process, and have an exponential distribution with mean 1 kilobyte.

The router maintains two parallel queues, a 'payload queue' and a 'header queue'. When a packet arrives, the payload is stored in the former, and a packet header is stored in the latter. Packets are served in the order they arrive. The payload queue is served at constant rate $C$ kilobytes per second, and when the entire payload of a packet has been served then that packet's header is removed from the header queue. Assume $C > \lambda$.

Both queues have finite space. The payload queue has space for 1000 kilobytes; the header queue has space for 1000 headers. As a queueing theorist, you are called in to advise on whether these are sensible choices.

(a) Let $Q$ be the number of packet headers in the header queue. With reference to an $M/M/1$ queue (or otherwise), estimate the probability that $Q \geqslant q$. (For modelling purposes, you can treat both queues as having infinite space.)

(b) The payload queue may be modelled by a discrete-time queue, with timeslots of length $\delta$, in which the number of packets arriving in each timeslot is a Poisson random variable with mean $\delta\lambda$, and the service offered in that timeslot is $C\delta$. Let $R_\delta$ be the amount of work in this discrete-time queue. Estimate the probability that $R_\delta \geqslant r$. (Again, for modelling purposes, you can treat both queues as having infinite space.)

(c) Which queue is more likely to overflow? Give an intuitive explanation for your answer.

*Hint. If $N$ is a Poisson random variable with mean $\lambda$ then $\mathbb{E}t^N = e^{\lambda(t-1)}$. If $X$ is an exponential random variable with mean $\lambda^{-1}$ then $\mathbb{E}e^{\theta X} = \lambda/(\lambda - \theta)$.*

**4** Consider a queue operating in continuous time, with constant service rate $C$ and finite buffer $B$, with arrival process $a \in \mathcal{C}_\mu$. It is known that if $\mu < C$ then the queue size at time 0 may be written as

$$\bar{q}(a) = \sup_{t \geqslant 0} \Big\{ \Big( \sup_{0 \leqslant s \leqslant t} x(-s, 0] \Big) \wedge \Big( B + \inf_{0 \leqslant s \leqslant t} x(-s, 0] \Big) \Big\}$$

where $x(-s, 0] = a(-s, 0] - Cs$ and $x \wedge y = \min(x, y)$. When $B = \infty$, denote this function by $q$. It is also known that $\bar{q}$ and $q$ are continuous on $(\mathcal{C}_\mu, \|\cdot\|)$.

Suppose that $(A^L,\ L \in \mathbb{N})$ satisfies a large deviations principle in $(\mathcal{C}_\mu, \|\cdot\|)$ with good rate function

$$I(a) = \begin{cases} \int_{-\infty}^{0} \Lambda^*(\dot{a}_s)\, ds & \text{if } a \text{ is absolutely continuous} \\ \infty & \text{otherwise} \end{cases}$$

for some strictly convex rate function $\Lambda^*$ with $\Lambda^*(\mu) = 0$.

(a) Write down a large deviations principle for $q(A^L)$; let it have rate function $J$. Also write down a large deviations principle for $\bar{q}(A^L)$; let it have rate function $\bar{J}$.

(b) Show that $\bar{q}(a) \leqslant q(a)$. Hence (or otherwise) show that

$$\bar{J}(x) \geqslant \inf_{y \geqslant x} J(y).$$

(c) Show that $J$ is increasing. Deduce that $\bar{J}(x) \geqslant J(x)$.

(d) Let $x \leqslant B$. Show that $\bar{J}(x) \leqslant J(x)$. *Hint. Let $\hat{a}$ be the most likely path to attain $q(a) = x$. What is $\bar{q}(\hat{a})$?*

(e) Deduce that $\bar{q}(A^L)$ satisfies a large deviations principle with good rate function

$$\bar{J}(x) = \begin{cases} J(x) & \text{if } x \leqslant B \\ \infty & \text{otherwise.} \end{cases}$$

*Note. You may assume standard results about queues with infinite buffers.*