UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Tuesday 1 June, 2004   9 to 11

POISSON PROCESSES

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain carefully what is meant by saying that the random sequence $(p_1, p_2, p_3, \ldots)$ has the Poisson-Dirichlet distribution with parameter θ. Show that, for any polynomial ϕ with $\phi(0) = 0$,

$$\mathbb{E}\left\{\sum_{n=1}^{\infty} \phi(p_n)\right\} = \theta \int_0^1 \phi(x) x^{-1}(1-x)^{\theta-1} dx\,.$$

What does this tell you about the distribution of p_1?

2 The positions of trees in a large forest can be modelled as a Poisson process Π of constant density λ on $\mathbb{R}^2$. Each tree produces a random number of seeds having a Poisson distribution with mean μ. Each seed falls to earth at a point uniformly distributed over the circle of radius r whose centre is the tree. The positions of the different seeds relative to their parent tree, and the numbers of seeds produced by a given tree, are independent of each other and of Π. Prove that, conditional on Π, the seeds form a Poisson process Π^* whose mean measure depends on Π. Is the unconditional distribution of Π^* that of a Poisson process? Justify your answer.

3 A uniform Poisson process Π in the unit ball of $\mathbb{R}^3$ is one whose measure is Lebesgue measure (volume) on $B = \{(x, y, z) \in \mathbb{R}^3 \ ; \ r^2 = x^2 + y^2 + z^2 \leqslant 1\}$. Show that

$$\Pi_1 = \{r \ ; \ (x, y, z) \in \Pi\}$$

is a Poisson process on $[0, 1]$ and find its mean measure. Show that

$$\Pi_2 = \{(x/r, y/r, z/r) \ ; \ (x, y, z) \in \Pi\}$$

is a Poisson process on the boundary of B, whose mean measure is a multiple of surface area. Are Π_1 and Π_2 independent processes? Justify your answer.