

M. PHIL. IN STATISTICAL SCIENCE

Thursday 27 May 2004 1.30 to 3.30

PROBABILITY

*Attempt **THREE** questions.*

*There are **five** questions in total.*

The questions carry equal weight.

You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.

1 You throw $6n$ dice at random.

- a) Find the probability that each number appears exactly n times.
- b) Using Stirling's formula, show that this probability is approximately $Cn^{-5/2}$ for some constant C to be found.

2 State and prove Chebyshev's inequality.

State and prove the first Borel-Cantelli lemma.

Let X_1, X_2, \dots , be independent identically distributed random variables with $\mathbb{E}X = a$ and $\text{Var}X = \sigma^2 < \infty$. Denote $S_n = \sum_{j=1}^n X_j$. Show that $N^{-2} S_{N^2} \rightarrow a$ (a.s.) as $N \rightarrow \infty$.

3 Suppose that S and T are independent exponential random variables with means $1/\alpha$ and $1/\beta$ respectively.

- a) Find the distribution of $\min\{S, T\}$.
- b) Find the probability that $S \leq T$.
- c) Show that the two events $\{S \leq T\}$ and $\{\min\{S, T\} \geq t\}$ are independent.

4 Let X_1, X_2, \dots , be a sequence of independent identically distributed random variables with $\Pr(X = 1) = p = 1 - \Pr(X = -1)$ and let S_n be the generated asymmetric simple random walk starting at 0, i.e., $S_n = \sum_{j=1}^n X_j$. Suppose $p > 1/2$ and let $\phi(x) = \left(\frac{1-p}{p}\right)^x$.

- a) Show that $\phi(S_n)$ is a martingale.
- b) Denote $T_x = \inf\{n : S_n = x\}$. State the Optional Stopping Theorem and use it to show that for $a < 0 < b$,

$$\Pr(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}.$$

- c) For $a < 0$, show that $\Pr(\min_n S_n \leq a) = \Pr(T_a < \infty) = \left(\frac{1-p}{p}\right)^{-a}$.

5 The Poisson process of rate λ is the Markov chain $(N_t)_{t \geq 0}$ on $\mathbb{Z}_+ = \{0, 1, \dots\}$ with generator $q_{ii} = -\lambda$, $q_{i, i+1} = \lambda$.

a) Consider the process starting at 0 and define, for each j , $p_j(t) = \Pr_0(N_t = j)$. Show that the forward equations imply

$$p'_0(t) = -\lambda p_0(t), \quad p'_j(t) = -\lambda p_j(t) + \lambda p_{j-1}(t), \quad j \geq 1.$$

Solve this system, thus proving that if $N_0 = 0$ then N_t has Poisson distribution with parameter λt .

b) Let $(N_t^{(1)})_{t \geq 0}$ and $(N_t^{(2)})_{t \geq 0}$ be two independent Poisson processes of rates λ_1 and λ_2 respectively and starting at 0. Find the distribution of $N_t = N_t^{(1)} + N_t^{(2)}$.

c) Arrivals of the No. 1 bus form a Poisson process of rate 1 bus per hour, and arrivals of the No. 7 bus form a Poisson process of rate 7 buses per hour. What is the probability that exactly 3 buses pass by in one hour?