

M. PHIL. IN STATISTICAL SCIENCE

Friday 30 May 2003 1.30 to 3.30

PAPER 31

POISSON PROCESSES

*Attempt **TWO** questions.*

*There are **three** questions in total.*

The questions carry equal weight.

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Seeds are planted in a field $S \subset \mathbb{R}^2$. The random way they are sown means that they form a Poisson process on S with density $\lambda(x, y)$. The seeds grow into plants that are later harvested as a crop, and the weight of the plant at (x, y) has mean $m(x, y)$ and variance $v(x, y)$. The weights of different plants are independent random variables. Show that the total weight W of all the plants is a finite random variable with mean

$$\iint_S m(x, y) \lambda(x, y) \, dx dy$$

and variance

$$\iint_S \{m(x, y)^2 + v(x, y)\} \lambda(x, y) \, dx dy,$$

so long as these integrals are finite.

[Any general theorems you use must be clearly stated, but should not be proved.]

2 A Poisson process Π on the interval $S = (-1, 1)$ has the density

$$\lambda(x) = (1+x)^{-2}(1-x)^{-3}.$$

Show that Π has, with probability 1, infinitely many points in S , and that they can be labelled in ascending order as

$$\dots X_{-2} < X_{-1} < X_0 < X_1 < X_2 < \dots$$

with

$$X_0 < 0 < X_1.$$

Show that there is an increasing function $f : S \rightarrow \mathbb{R}$ with $f(0) = 0$ such that the points $f(X)$ ($X \in \Pi$) form a Poisson process of unit rate on \mathbb{R} , and use the strong law of large numbers to show that, with probability 1,

$$\lim_{n \rightarrow +\infty} (2n)^{1/2} (1 - X_n) = \frac{1}{2}.$$

Find a corresponding result as $n \rightarrow -\infty$.

3 A line L in \mathbb{R}^2 not passing through the origin O can be defined by its perpendicular distance $p > 0$ from O and the angle $\theta \in [0, 2\pi)$ that the perpendicular from O to L makes with the x -axis. Explain carefully what is meant by a Poisson process of such lines L .

A Poisson process of lines L has mean measure μ given by

$$\mu(B) = \iint_B dp \, d\theta$$

for $B \subseteq (0, \infty) \times [0, 2\pi)$. A random countable set $\Phi \subset \mathbb{R}^2$ is defined to consist of all intersections of pairs of lines in Π . Show that the probability that there is at least one point of Φ inside the circle with centre O and radius r is less than

$$1 - (1 + 2\pi r)e^{-2\pi r}.$$

Is Φ a Poisson process? Justify your answer.