UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Thursday 29 May 2003 9 to 12

PAPER 29

ADVANCED PROBABILITY

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $\mathcal{G}$ be a σ-algebra, $\mathcal{G} \subseteq \mathcal{F}$.

a) Let X, Y be random variables such that $Y = \mathsf{E}(X \mid \mathcal{G})$ and $\mathsf{E}X^2 = \mathsf{E}Y^2 < \infty$. Show that $Y = X$ almost surely.

b) Let X be a random variable such that $\mathsf{E}X^2 < \infty$ and $\mathsf{E}(X \mid \mathcal{G})$ has the same distribution as X. Show that $X = \mathsf{E}(X \mid \mathcal{G})$ almost surely.

c) Let X be a random variable such that $\mathsf{E}|X| < \infty$ and $\mathsf{E}(X \mid \mathcal{G})$ has the same distribution as X. Show that $X = \mathsf{E}(X \mid \mathcal{G})$ almost surely.

[Hint: Show that $\operatorname{sign}(X) = \operatorname{sign}\big(\mathsf{E}(X \mid \mathcal{G})\big)$ almost surely, where

$$\operatorname{sign}(a) = \begin{cases} 1, & a > 0, \\ 0, & a = 0, \\ -1, & a < 0. \end{cases}$$

]

2 State and prove the almost sure martingale convergence theorem.

Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration in $(\Omega, \mathcal{F}, \mathsf{P})$ and let X_n, $Y_n > 0$ be integrable and adapted to $\mathcal{F}_n$. Suppose that for any $n \geq 0$

$$\mathsf{E}\big(X_{n+1} \mid \mathcal{F}_n\big) \leq (1 + Y_n) X_n$$

with $\sum_{n \geq 0} Y_n < \infty$ almost surely. Show that X_n converges almost surely to a finite limit as $n \to \infty$.

3 Let B be a Brownian motion starting from 0 and T_a be the hitting time of $a > 0$, $T_a = \inf\{t \geq 0 : B_t = a\}$. Let $\varphi_a(\lambda)$ be the Laplace transform of T_a,

$$\varphi_a(\lambda) = \mathsf{E}\exp\{-\lambda T_a\}, \qquad \lambda \geq 0.$$

a) Show that for all x, $y > 0$

$$\varphi_x(\lambda)\,\varphi_y(\lambda) = \varphi_{x+y}(\lambda)$$

and thus $\varphi_a(\lambda) = \exp\{-ac(\lambda)\}$ for some $c(\lambda) \geq 0$. Show also that the scaling property of B implies that $c(\lambda) = C\sqrt{\lambda}$ with some constant $C \geq 0$.

b) Let $b \leq 0$ and $\tau = \inf\{t \geq 0 : B_t = a + bt\}$ with $a > 0$. Using the exponential martingale $\exp\{\theta B_t - \theta^2 t/2\}$ or otherwise, show that

$$\mathsf{E}\exp\{-\lambda\tau\} = \exp\{-a(b + \sqrt{b^2 + 2\lambda})\}, \qquad \lambda > 0.$$

Deduce that $\varphi_a(\lambda) = \exp\{-a\sqrt{2\lambda}\}$.

4 a) Define carefully weak convergence of probability measures and discuss its characterization in terms of characteristic functions. State carefully, and prove, the corresponding theorem for probability measures on the real line.

b) Let $(X_j)_{j\geq 1}$ be a sequence of independent random variables with common distribution

$$\mathsf{P}(X = +1) = \mathsf{P}(X = -1) = \tfrac{1}{2}.$$

For each $n \geq 1$, define $Y_n = \sum_{j=1}^{n} 2^{-j} X_j$.

(1) Find the distribution of Y_n, show that it converges weakly as $n \to \infty$ and identify the corresponding limit.

(2) Show that Y_n also converges almost surely and in L^2.

(3) Give a probabilistic proof of the equality

$$\frac{\sin t}{t} = \prod_{j=1}^{\infty} \cos\frac{t}{2^j}, \qquad t \in \mathbb{R},$$

where we put $\frac{\sin 0}{0} = 1$.

[Hint: Recall part a) of the question.]

5 What is meant by saying that Brownian motion in $\mathbb{R}^2$ is neighbourhood recurrent but does not hit points? State carefully, and prove, the corresponding theorem.

6 a) Suppose X_n^1 and X_n^2 are supermartingales with respect to filtration $\big(\mathcal{F}_n\big)_{n\geq 0}$ and N is a stopping time with the property that $X_N^1 \geq X_N^2$. Show that

$$Y_n = X_n^1\, \mathbb{1}_{\{N>n\}} + X_n^2\, \mathbb{1}_{\{N\leq n\}}$$

is a supermartingale with respect to $\big(\mathcal{F}_n\big)_{n\geq 0}$.

b) Let $\big(X_n\big)_{n\geq 0}$ be a positive supermartingale. For fixed a, b, $b > a > 0$, we put $N_0 = -1$ and for $j \geq 1$ let

$$N_{2j-1} = \inf\big\{m > N_{2j-2} : X_m \leq a\big\}, \quad N_{2j} = \inf\big\{m > N_{2j-1} : X_m \geq b\big\}.$$

Denote by U the number of upcrossings of $[a,b]$. Let $Y_n = 1$ for $0 \leq n < N_1$ and for $j \geq 1$

$$Y_n = \begin{cases} \big(b/a\big)^{j-1} X_n/a, & N_{2j-1} \leq n < N_{2j}, \\ \big(b/a\big)^{j}, & N_{2j} \leq n < N_{2j+1}. \end{cases}$$

(1) Show that $Y_{n\wedge N_j}$ is a supermartingale for any $j \geq 0$.

(2) Deduce that the number U of upcrossings of $[a,b]$ satisfies Dubins' inequality:

$$\mathsf{P}\big(U \geq k\big) \leq \big(a/b\big)^k \mathsf{E}\min\big(X_0/a, 1\big).$$