UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Wednesday 6 June 2001 9 to 12

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on optimal hedging in the least-squares sense in a one-period financial model. Your essay should cover the notions of attainable claims, dominated and equivalent martingale measures, the minimal martingale measure and a proof of the fact that the model is complete if and only if there is a unique dominated martingale measure.

2 '*The value of any portfolio is the conditional expected value under any martingale measure of the sum of its future discounted dividends.*'

Write an essay which explains this statement in the context of a multiperiod, discrete-time market.

3 State and prove Girsanov's Theorem. Describe briefly the relevance of this result to the pricing of contingent claims in the Black-Scholes model.

Let $W_t^\mu = W_t + \mu t$ denote a standard Brownian motion with linear drift and set $M_t^\mu = \sup_{0 \leqslant s \leqslant t} W_s^\mu$. Prove that for $a > 0$ and $x \leqslant a$,

$$\mathbb{P}\left(W_t^\mu \leqslant x,\, M_t^\mu < a\right) = \Phi\left(\frac{x - \mu t}{\sqrt{t}}\right) - e^{2a\mu}\Phi\left(\frac{x - 2a - \mu t}{\sqrt{t}}\right).$$

Consider an option that pays 1 at time t_0 if the stock price $\{S_t\}$ reaches a fixed level $b > S_0$ at some time in the interval $[0, t_0]$; otherwise, it pays nothing. In the context of the Black-Scholes model, derive an expression for the price at time 0 of this option.

4 Suppose that in the Black-Scholes model, the stock price at time t is S_t, the fixed interest rate is ρ and the volatility is σ. Show that $f(S_t, t)$ is the value of a self-financing portfolio if and only if the function $f = f(x, t)$ satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \rho x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} - \rho f = 0.$$

Determine the price at time $t < t_0$ of a European-style claim that pays $C = S_{t_0} \ln(S_{t_0})$ at time t_0. What is the holding in stock at time t in the portfolio that replicates this claim? Justify your answer.

5 **Either**: Write an essay on determining optimal investment strategies for an investor in the Black-Scholes model who seeks to maximize the expected utility of his final wealth at time $t_0 > 0$ for given initial wealth at time 0.

or: Write an essay on pricing path-dependent contingent claims in the Black-Scholes model.

6 Write an essay on the modelling of interest rates using Gaussian random fields.