UNIVERSITY OF CAMBRIDGE

# M. PHIL. IN STATISTICAL SCIENCE

Friday 1 June 2001 9 to 11

## PROBABILITY

*Attempt* ***three*** *of the following five questions.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) What does it mean to say that $(X_n)_{n \geqslant 0}$ is a discrete-time Markov chain with state space $S$ and transition matrix $P$?

(b) State the strong Markov property of such a process $(X_n)_{n \geqslant 0}$.

(c) Fix a state $i$ and let $f_i$ denote the return probability for $i$, that is, the probability when $X_0 = i$, that $X_n = i$ for some $n \geqslant 1$. Show that, if $f_i = 1$, then $X_n$ visits $i$ infinitely often, or not at all.

Show also that, if $f_i < 1$, then $X_n$ visits $i$ only finitely often.

(d) Consider the Markov chain $(X_n)_{n \geqslant 0}$ in $\mathbb{Z}$ whose non-zero transition probabilities are given by

$$P_{0,-1} = 1, \quad P_{i,i-1} = \frac{1}{4}, \quad P_{i,i+1} = \frac{3}{4}, \quad i \neq 0.$$

If $X_0 = 1$, what is the probability that $X_n$ visits 0 infinitely often.

**2** Let $X, X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose that

$$\mathbb{E}(X) = \mu, \quad \mathbb{E}(X^4) < \infty.$$

(a) Show that

$$\frac{X_1 + \ldots + X_n}{n} \to \mu \quad a.s. \ \text{ as } \ n \to \infty$$

(b) Suppose that $\mu = 0$. Is it true that $(X_1 + \ldots + X_n)/\sqrt{n}$ converges in distribution as $n \to \infty$? Is it true that $(X_1 + \ldots + X_n)/\sqrt{n}$ converges in $L_2(\mathbb{P})$? Justify your answers by reference to standard theorems.

**3** Let $(X_n)_{n \geqslant 0}$ be a Markov chain with state-space $S$ and transition matrix $P$.

(a) Write $x \to y$ if $P^n_{x,y} > 0$ for some $n \geqslant 0$ and write $x \sim y$ if both $x \to y$ and $y \to x$. Show that $\sim$ is an equivalence relation on $S$.

(b) Assume that $\sim$ has just one equivalence class and that $P$ has an invariant distribution $\pi$. Suppose there exists a bijection $T : S \to S$ such that, for all $x, y \in S$,

$$P_{T(x),T(y)} = P_{x,y}.$$

Show that $\pi_{T(x)} = \pi_x$ for all $x \in S$.

(c) Let $p \in (0,1)$. Set $X_0 = I$ and suppose, for $n \geqslant 0$,

$$X_{n+1} = \begin{cases} AX_n, & \text{with probability } p, \\ BX_n, & \text{with probability } 1-p, \end{cases}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Sketch a graph of the state-space of the Markov chain $(X_n)_{n \geqslant 0}$, indicating the possible transitions. Determine whether $(X_n)_{n \geqslant 0}$ is transient, null recurrent or positive recurrent.

**4** Let $X, X_1, X_2, \ldots$ be independent, identically distributed random variables.

Set

$$M_X(t) = \mathbb{E}(e^{tX}),\ c_X(t) = \log M_X(t).$$

(a) Show, for $S_n = X_1 + \ldots + X_n$, for $t \geqslant 0$ and $x \in \mathbb{R}$, that

$$\mathbb{P}(S_n \geqslant nx) \leqslant e^{-n\{xt - c_X(t)\}}.$$

Show also that this inequality remains valid for $t < 0$, provided $x \geqslant \mathbb{E}(X)$.

Deduce that, for $x \geqslant \mathbb{E}(X)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{S_n}{n} \geqslant x) \leqslant -I_X(x),$$

where

$$I_X(x) = \sup_{t\in\mathbb{R}}\{xt - c_X(t)\}.$$

(b) Write $I_\lambda$ for $I_X$ when $X \sim \mathcal{E}(\lambda)$ and write $I_p$ for $I_x$ when $X \sim B(1,p)$. Compute $I_\lambda$ and $I_p$. Here, $\mathcal{E}(\lambda)$ denotes the exponential distribution of rate $\lambda$ and $B(n,p)$ denotes the binomial distribution with parameters $n$ and $p$.

(c) When fitting a light bulb, there is a chance of $1-p$ that it fails instantaneously. Given that it does not, its lifetime has $\mathcal{E}(\lambda)$ distribution. Let $p_n(x)$ denote the probability that the average lifetime of $n$ bulbs exceeds $x$, where $x > 1/\lambda$. Show that, for all $n$,

$$p_n(x) \leqslant \mathbb{E}(e^{-nF_x(N/n)})$$

where $N$ has $B(n,p)$ distribution and

$$F_x(y) = yI_\lambda(x/y).$$

Deduce that

$$\limsup_{n\to\infty} \quad \frac{1}{n}\log p_n(x) \leqslant -\inf_{y\in[0,1]}\{I_p(y) + yI_\lambda(x/y)\}.$$

**5** State Doob's $L_1$ and $L_2$ martingale convergence theorems.

Let $f : [0,1] \to \mathbb{R}$ and assume that for some $K > 0$, we have $|f(x) - f(y)| \leqslant K|x-y|$ for all $x$ and $y$. For every $n$, let $t_{i,n} = i/2^n$. Define $M_n : [0,1) \to \mathbb{R}$ by

$$M_n(x) = 2^n(f(t_{i+1,n}) - f(t_{i,n})), \quad t_{i,n} \leqslant x < t_{i+1,n}, \quad i = 0,1,\ldots,2^n-1.$$

Show that, for almost all $x$, the limit $M(x) = \lim_{n\to\infty} M_n(x)$ exists and satisfies

$$\int_0^1 M(x)dx = f(1) - f(0).$$

What is $M$ better known as?