

Mathematical Tripos

Part III Lecture Courses in 2011-2012

Department of Pure Mathematics
& Mathematical Statistics

Department of Applied Mathematics
& Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is *no* requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2011-12. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures define the syllabus. Each course lecturer has discretion to vary the material covered.
- This document was last updated in September 2011. Further changes to the list of courses will be avoided if at all possible, but may be essential.
- Some graduate courses have no writeup available. Hopefully, the title of the course is sufficiently explanatory.

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Algebra

Lie Algebras and their representations (M24)

C.J.B. Brookes

While groups arise from the study of symmetries, Lie algebras concern infinitesimal symmetries. They are non-associative algebraic structures arising in many branches of mathematics and theoretical physics.

The core material for this course is the theory of finite dimensional Lie algebras and their finite dimensional representations. This involves some basic structure theory: nilpotent, soluble and semisimple Lie algebras, the Killing form, derivations, Borel and Cartan subalgebras, Weyl groups. The classification of finite dimensional complex semisimple Lie algebras arises from considering finite root systems and their classification using Dynkin diagrams.

An introduction to the representation theory of semisimple Lie algebras starts with the consideration of weights and highest weight modules, of which Verma modules are a special case. Depending on how time is flying I will discuss the universal enveloping algebra which is the associative algebra underlying the representation theory of Lie Algebras. In general for finite dimensional Lie algebras the enveloping algebra can be viewed as a non-commutative polynomial algebra.

Time permitting I may also say something about Lie algebras in characteristic p .

Pre-requisite Mathematics

To develop the theory of Lie algebras one just needs basic linear algebra. But for the representation theory it would certainly be helpful, but not absolutely essential, to have some experience of representation theory including some module theory.

Books

1. Dixmier J, Enveloping algebras North Holland (1977)
2. Erdmann K, Wildon MJ, Introduction to Lie algebras, Springer (2006)
3. Humphreys JE, Introduction to Lie algebras and representation theory, Springer (1972)
4. Jacobson N, Lie algebras, Dover (1979)
5. Serre J-P, Complex semisimple Lie algebras, Springer (1987)

Stable commutator length. (L24)

Non-Examinable (Graduate Level)

Danny Calegari

This is a graduate course focussed on an important subtopic in geometric group theory. In this course we present and discuss some elements of the geometric theory of 2-dimensional (bounded) homology from several points of view, making contact with low-dimensional geometry and topology, with combinatorics, with ergodic theory and probability, and with dynamics.

The main topics to be covered will include

1. Generalized Bavard duality
2. Hyperbolicity and spectral gaps

3. Free and surface groups
4. Transformation groups and symplectic geometry
5. Statistical properties

We hope to get to a discussion of a number of open problems which might make good thesis topics.

Desirable Previous Knowledge

The course will be self-contained, but requires mathematical maturity. There are few prerequisites for the class, since much of the material will be presented from scratch. Students should know about the topology of surfaces, some of the details of a first course in algebraic topology, and perhaps some hyperbolic geometry (though this will not be necessary if the student is prepared to take a couple of things on faith). Some background in probability would be helpful.

Introductory Reading

Although it will not be strictly necessary, students are encouraged to glance at one or two background references in low dimensional geometry/topology and geometric group theory. The following references are “classic”; the student is **not** expected to have mastered them, or even to have skimmed them.

1. Gromov, M. Asymptotic invariants of infinite groups. LMS Lecture Note Series, 182 (1993)
2. R. Benedetti and C. Petronio. Lectures on hyperbolic geometry. Springer Universitext (2003)
3. Thurston, W. Three dimensional geometry and topology. Princeton Math. Series, 35 (1997)

Reading to complement course material

The textbook for the course is my monograph, which can be freely downloaded from my webpage, or purchased, from Amazon or directly from World Scientific (you are encouraged to purchase a copy if you find it useful). Some more advanced material not covered in this textbook is contained in papers, generally available on the arXiv.

1. Calegari, D. scl. Mathematical Society of Japan Monographs, 20 (2009)

Topics in Calculus and Algebra. (L24)

I. Grojnowski

This course is an introduction to basic homological algebra and deformation theory as it occurs in algebraic geometry, topology and physics.

I’ll choose one or two topics that are currently of interest, and develop them, explaining the basic examples as well as the necessary machinery surrounding them.

One possibility is that we’ll study the (infinity, 2) category of sheaves of DG categories on an algebraic variety, beginning with the most basic definitions. The papers of Toen, and books of Lurie, are the key references.

We might also study the fundamental work of Kontsevich on the B-model, and the Tamarkin-Tsygan idea of “calculus”. (The somewhat flamboyant title is theirs—it refers to the structures for arbitrary algebras that generalise the differential forms/vector fields package that are attached to smooth manifolds.)

There will be no particular prerequisites, other than the basic courses.

Representation theory and invariant theory (L24)

Stuart Martin

This is a lecture course on the representation theory of the symmetric group, the general linear group and some related invariant theory. My aim is to cover as much of the beautiful classical theory as time permits, so, for example, I will usually restrict myself to working over the complex numbers.

For the symmetric group: a review of semisimple algebras; Young symmetrizers, Young tableaux and diagram calculus; Specht modules, character computations; the Hook Formula for dimensions of the Specht module.

For the general linear group: a review of multilinear algebra; Schur-Weyl duality; the decomposition of tensors; rational and polynomial representations of $GL(V)$; Weyl's character formula.

Invariant theory: basic examples, symmetric polynomials; the First Fundamental Theorem of invariant theory; (if time permits) Gordan's theorem and the computation of covariants.

Pre-requisite Mathematics

No previous knowledge is required beyond a smattering of rings and modules, ordinary character theory, and affine varieties.

Level

Part III

Literature

1. D.J. Benson, Polynomial invariants of finite groups. CUP 1993.
2. J.A. Dieudonné and J.B. Carrell. Invariant theory old and new. Academic Press 1971.
3. I.M. Isaacs, Character theory of finite groups. AMS Chelsea Publishing.
4. J.H. Grace and A. Young, The algebra of invariants. CUP 1903 (plus subsequent reprints). A classic.
5. G.D. James and A. Kerber, The representation theory of the symmetric group. CUP 1984.
6. I.G. Macdonald. Symmetric functions and Hall polynomials (2nd edn.) OUP 1995.
7. H. Weyl, The classical groups: their invariants and representations. Princeton U.P. 1946 (plus subsequent reprints).

Computational Group Theory (M24)

Non-Examinable (Graduate Level)

R.A. Parker

Computational Group Theory is the study of ways of using a computer to solve problems in group theory, though in fact this is a thriving and active area at present, frequently posing questions that lead, rather than follow, theoretical work.

This course needs little preparation above an undergraduate level of algebra and some knowledge of computing.

The course will first cover much of the "classical" computational group theory, including the various algorithms for working with permutations, rewriting (which will also include Groebner Bases) and working with matrices over finite fields. This part will encompass about the first twelve lectures.

The second half of the course will cover the more recent work, including working with matrices of integers (to represent the group, or otherwise), the (very) current Matrix recognition project (what group do these matrices generate?) and Small Cancellation methods.

No preliminary reading is required, but the following two books present (between them) a considerable amount of the course.

1. Computation with Finitely Presented Groups Charlie C Sms C.U.P. 1994
2. Handbook of Computational Group Theory Derek F Holt, Bettina Eick and Eamonn OBrien Chapman+Hall/CRC 2005

Commutative Algebra (M24)

B. Totaro

Commutative algebra is the study of commutative rings, the basic examples being the ring of integers \mathbf{Z} and the ring of polynomials $k[x_1, \dots, x_n]$ over a field. These examples correspond to the two main subjects where commutative algebra is used: number theory and algebraic geometry. Commutative algebra is also fundamental for algebraists. My interests lie in algebraic geometry, and one aim of the course is to prove the algebraic results used in Part III Algebraic Geometry.

We try to think of any commutative ring as the ring of functions on some space, so that there is a constant interplay between algebra and geometry. In that spirit, the course starts by defining the prime spectrum of a ring, which is a topological space. Localization of rings and modules is an algebraic notion that corresponds to working in a neighborhood of a point. We define noetherian rings, which include all the rings we are concerned with. We prove the Hilbert basis theorem, the Hilbert Nullstellensatz, and the Noether normalization lemma: these are foundational results for algebraic geometry. We study Dedekind domains and discrete valuation rings, which are the well-behaved rings of dimension 1.

We define regular local rings, which correspond in geometric terms to smooth varieties (or manifolds). We develop the rich theory of dimension for commutative rings. Integral closure is an algebraic concept which describes both the arithmetic notion of algebraic integers and the geometric process of “normalization”.

For much of the course, our approach will be close to the beautiful short book by Atiyah and Macdonald. That corresponds to about the first half of Eisenbud’s book. Compared to Atiyah-Macdonald, we include more homological algebra (flat modules, projective modules, Ext and Tor), and we prove some of the deeper results on regular local rings.

Pre-requisite Mathematics

Undergraduate algebra is the main background needed (especially fields, rings, and modules). Lang’s *Algebra* is a good graduate-level reference that includes all the background needed, as well as some of the course material.

Literature

1. M. Atiyah and I. Macdonald. *Introduction to commutative algebra*. Westview (1994), £42.99.
2. D. Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Springer (1995), £27.19.
3. S. Lang. *Algebra*. Springer (2002), £53.99.
4. M. Reid. *Undergraduate commutative algebra*. Cambridge (1993), £20.39.

Analysis

Calculus of Variations (L24)

Non-Examinable (Graduate Level)

Sophia Demoulini

The direct method. Minimizing sequences, weak compactness, Euler-Lagrange equations. Necessary conditions; sufficient conditions.

Convexity, quasiconvexity and (weak) lower semicontinuity; polyconvex and rank-one convex envelopes. Non-convex problems and the Young measure.

Examples, including harmonic maps and polyconvex elastic functionals, Skyrmions.

Introduction to Functional Analysis. (M24)

Dr D.J.H. Garling

This is a basic course on functional analysis. The material is abstract, rather than practical, and most of the results are very general. But I shall concentrate on topics and results which are widely used in analysis. The course will include some or all of the following topics.

Filters, ultrafilters and Tychonoff's theorem.

Convexity, and the Hahn-Banach theorem. Banach space duality.

The weak*-topology, and the Banach-Alaoglu theorem.

Measures on compact spaces.

The Riesz representation theorem, on the dual of $C(K)$.

Von Neumann's proof of the Radon-Nikodym theorem.

Haar measure on compact groups.

Banach space fixed point theorems.

Extreme points. The Krein-Mil'man theorem, and Choquet theory.

Introduction to Banach algebras

Desirable Previous Knowledge

Basic knowledge of analysis and general topology. Results from the the Part II Linear Analysis course, such as the self duality of Hilbert space, the Stone-Weierstrass theorem and the Ascoli-Arzelà theorem will be used, as will results from the Part II Probability and Measure course, such as Carathéodory's extension theorem. But detailed knowledge of their proofs will not be needed.

Introductory Reading

1. B. Bollobas. Linear Analysis. CUP 1990.
2. R.M.Dudley. Real analysis and probability (Chapters 1-5). CUP 2002.

Reading to complement course material

1. G.R. Allan. Introduction to Banach spaces and algebras. OUP 2011.
2. B. Bollobas. Linear Analysis. CUP 1990.
3. R.M.Dudley. Real analysis and probability. CUP 2002.
4. W.Rudin. Functional analysis. McGraw-Hill. 1973-1990.
5. B.Simon. Convexity; an analytic viewpoint. CUP 2011.
6. P. Wojtaszczyk. Banach spaces for analysts. CUP1991.

Elliptic Partial Differential Equations (M24)

B Kummel

This course is intended as an introduction to the theory of linear elliptic partial differential equations. Elliptic equations play an important role in geometric analysis and a strong background in linear elliptic equations provides a foundation for understanding other topics including minimal submanifolds, harmonic maps, and general relativity. The course will begin by introducing elliptic equations via harmonic functions. We will then proceed to consider classical and weak solutions to elliptic equations, studying the maximum principle, Schauder estimates, the existence and uniqueness of solutions to the Dirichlet problem, and the regularity of solutions. As time permits, we will discuss the De Giorgi-Nash theory, which can be used to prove the Harnack inequality and establish Hölder continuity for weak solutions. Note that the basic properties of Sobolev spaces that we will need for weak solutions will be covered as part of the course.

Prerequisites

Lebesgue integration, Lebesgue spaces, and basic functional analysis.

Literature

1. David Gilbarg and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).
2. Lawrence Evans, Partial Differential Equations. AMS (1998)
3. Qing Han and Fanghua Lin, Elliptic partial differential equations. Courant Lecture Notes, Vol. 1 (2011).

Mathematical Topics in Kinetic Theory (M24)

Clément Mouhot

This is an introductory course to the modern mathematical analysis of the so-called kinetic equations. These nonlinear partial differential equations (PDEs) are derived from statistical physics. Their mathematical study has been extremely and increasingly active in the last decades. The final goal of this course is to present some results on the historically fundamental Boltzmann equations in gas dynamics and/or Vlasov-Poisson equations in plasma physics and galactic dynamics. We shall meanwhile review a range of topics in PDE analysis needed to this purpose.

The general plan of the course should include part or all of the following:

1. Introduction
 - 1.1. From ODEs to PDEs

- 1.2. Microscopic Viewpoint and Historical Comments
- 1.3. Statistical Viewpoint: Kinetic Theory
- 1.4. Mean Field Limits and Vlasov Models
- 1.5. Boltzmann-Grad Limit and Collisional Models
2. Linear Kinetic Equations
 - 2.1. The Characteristics Method: *solutions, invariants, maximum principle*
 - 2.2. The Linear Transport Equation: *dispersion estimates, averaging lemma, phase mixing*
 - 2.3. The Linear Boltzmann Equation: *entropy structure, hypocoercivity, trend to equilibrium*
3. The Vlasov-Poisson Equations
 - 3.1. Fundamental a Priori Estimates
 - 3.2. Construction of Weak Solutions
 - 3.3. Propagation of Moments and Regularity
 - 3.4 Further Comments: *mean-field limit and derivation of the equation, Landau damping*
4. The Boltzmann Equation
 - 4.1. Fundamental a Priori Estimates
 - 4.2. Construction of Spatially Homogeneous Solutions
 - 4.3. Construction of Close-to-equilibrium Solutions
 - 4.4. The DiPerna-Lions Renormalized Solutions
 - 4.5. Further Comments: *Trend to equilibrium (2d principle of thermodynamics), Hilbert's 6th problem (hydrodynamical limit, Boltzmann-Grad limit)*

Desirable previous knowledge

Important prerequisites: Measure theory, Lebesgue spaces and basics in functional analysis.

Useful but non compulsory previous knowledge: Sobolev spaces and Sobolev inequalities; basics in PDEs; basics in semigroups theory. **Introductory Reading**

- Cercignani, C.: The Boltzmann equation and its applications. Applied Mathematical Sciences, 67. Springer-Verlag, New York, 1988.
- Dautray, R., Lions, J.-L.: Mathematical analysis and numerical methods for science and technology. Vol. 6. Evolution problems. II. Springer-Verlag, Berlin, 1993.
- Cercignani, C., Illner, R., Pulvirenti, M.: The mathematical theory of dilute gases. Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.
- Glassey, R. T.: The Cauchy problem in kinetic theory. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
- Villani, C.: A review of mathematical topics in collisional kinetic theory. Handbook of mathematical fluid dynamics, Vol. I, 71–305, North-Holland, Amsterdam, 2002.

Reading to complement course material

Here are some further readings for complements in physics, recalls in analysis, or examples of material on more specific topics related to the course.

- Lifshitz E. M., Pitaevskii L.P.: Physical Kinetics (Course of Theoretical Physics “Landau-Lifshitz”, vol. 10), Butterworth-Heinemann, 1981.
- Lieb, E. H., Loss, M.: Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
- Brilliantov, N. V., Pshchel, T.: Kinetic theory of granular gases. Oxford Graduate Texts. Oxford University Press, Oxford, 2004.
- Golse, F.: The Boltzmann equation and its hydrodynamic limits. Evolutionary equations. Vol. II, 159–301, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005.

Kac–Moody and Virasoro Algebras (L24)

Antony Wassermann

This course will develop the representation theory of affine Kac–Moody algebras and the Virasoro algebra. These infinite–dimensional Lie algebras play an important rôle in string theory and conformal field theory; they are the Lie algebras of the loop groups and diffeomorphism group of the circle. We will adopt a unitary viewpoint and use supersymmetry as the main technique, as suggested by the supersymmetric coset constructions of Goddard–Kent–Olive and Kazama–Suzuki. Even in the case of finite–dimensional simple Lie algebras, this approach is fruitful. Not only does it give a streamlined route to the Weyl character formula, but it also has a natural geometric interpretation in terms of Dirac operators and index theory. A previous version of the course was given in Part III in 1998 and can be found here:

<http://arxiv.org/abs/1004.1287>

The current course will have an intersection with those lecture notes, and in particular will construct the finite–dimensional simple Lie algebras directly from their root lattices. However, the representation theory will be developed only for $sl(2)$ and its affine Kac–Moody algebra. These results will then be used to prove the character formula for the discrete series of the Virasoro algebra, using the easy part of the Friedan–Qiu–Shenker theorem. The course will include a gentle introduction to vertex algebras and boson–fermion duality, expanding part of the minicourse given at MSRI in 2000:

<http://www.msri.org/publications/ln/msri/2000/opal/wassermann/>

No prior knowledge of Lie algebras is required for this course.

1. P. Goddard and D. Olive, Kac–Moody and Virasoro algebras, World Scientific.
2. V. Kac, Infinite dimensional Lie algebras, C.U.P.
3. V. Kac and A. Raina, Bombay lectures on highest weight representations of infinite–dimensional Lie algebras, World Scientific.
4. V. Kac, Vertex Algebras for Beginners, University Lecture Note Series, A.M.S.
5. E. de Kerf, G. Bauerele and A. ten Kroode, Lie algebras, Studies in Mathematical Physics, North Holland.
6. A. Pressley and G. Segal, Loop groups, O.U.P.

Topics in Partial Differential Equations (L24)

Non-Examinable (Graduate Level)

Neshan Wickramasekera

The goal of this course is to introduce some general principles in modern geometric PDE by focusing on the PDE aspects of (one or both of) the following specific examples: (1) minimal submanifolds (2) harmonic maps between Riemannian manifolds. Both these classes of objects arise as solutions to variational problems; the first as critical points of the area functional and the second as critical points of the Dirichlet energy functional. Their analysis relies heavily on general techniques from linear and non-linear second order elliptic partial differential equations.

Desirable Previous Knowledge

Some basic knowledge of the theory of second order linear elliptic PDE as covered e.g. in the the part III course Elliptic Partial Differential Equations.

Reading to complement course material

1. Chapters 8 and 11 of Elliptic Partial Differential Equations of Second Order by D. Gilbarg and N. Trudinger.
2. Theorems on Regularity and Singularity of Energy Minimizing Maps by L. Simon
3. Curvature Estimates for Minimal Hypersurfaces by R. Schoen, L.Simon and S.-T. Yau, Acta Mathematica 134 (1975) 275–288.

Combinatorics

Percolation on Graphs. (L16)

Béla Bollobás

The late 1950s saw the foundation of two major areas of mathematics: percolation theory by Broadbent and Hammersley in 1957, and the theory of random graphs by Erdős and Rényi in 1959. Both theories concern random subgraphs of certain ground graphs: the former the random subgraphs of *infinite lattice-like graphs*, especially standard lattices such as \mathbb{Z}^d , the hexagonal lattice, or the triangular lattice, and the latter the random subgraphs of a *finite complete graph*.

The first basic question in percolation theory, proposed already by Broadbent and Hammersley in their founding paper, is whether our infinite random subgraph has an *infinite* component, and the most famous result of Erdős and Rényi concerns the existence of a *large* component in our finite random subgraph – large in terms of the number of vertices of the underlying complete graph.

In spite of these striking similarities, for many years these theories developed independently of each other: only for the past twenty years or so have they been considered to be facets of the same subject. In the course I shall present a number of results in the confluence of these theories.

The course will be accessible to people knowing a smattering of probability theory and combinatorics. There is no book that the course could follow, but some printed notes will be produced to help the participants.

Some Relevant Papers

B. Bollobás, S. Janson and O. Riordan, The phase transition in inhomogeneous random graphs, *Random Structures and Algorithms* **31** (2007), 3–122.

B. Bollobás and O. Riordan, Random graphs and branching processes, in *Handbook of Large-Scale Random Networks*, Bolyai Soc. Math. Stud. **18**, Springer, Berlin, 2009, pp. 15–115.

B. Bollobás, S. Janson and O. Riordan, The cut metric, random graphs, and branching processes, *J. Stat. Phys.* **140** (2010), 289–335.

Topological methods in combinatorics (L16)

Non-Examinable (Graduate Level)

Boris Bukh

Topology is an infrequent but a powerful visitor to the world of combinatorics. In this course we shall see applications to graph theory, combinatorial geometry, and complexity theory.

From topology we will cover simplicial complexes, Borsuk–Ulam theorem, ham-sandwich theorem (with cheese), nerve theorem, \mathbf{Z}_2 -index. Combinatorial applications will include necklace splittings, chromatic numbers of Kneser graphs, topological Radon’s theorem, evasiveness.

Prerequisite Mathematics

Exposure to algebraic topology is very helpful, but not required.

Reading material

We will mostly follow “Using Borsuk–Ulam theorem” by Jiří Matoušek.

Techniques in additive combinatorics (M16)

Non-Examinable (Graduate Level)

W. T. Gowers

In this course I shall aim to convey several of the main concepts and methods of additive combinatorics, an area that has grown very rapidly over the last fifteen years or so. Since it is a graduate course rather than a Part III course, I shall not feel obliged to give full details of all the proofs: instead, the lectures will be more like research seminars. The payoff for this is that I will (I hope) be able to cover much more ground, and the lectures will (I hope) be fairly easy to follow even for busy Part III students who do not have time to read their notes between lectures.

I shall begin by explaining some of the ideas that go into the proof of the Green-Tao theorem, which will require me to look at a number of other results, including Plunnecke’s inequality, Freiman’s theorem and Szemerédi’s theorem, on the way. After that, I shall see how much time is left. Possible further topics are Dvir’s solution to the finite-fields Kakeya problem, the hypergraph regularity and counting lemmas, the density Hales-Jewett theorem, proving combinatorial results inside sparse random sets, the random sampling method of Croot and Sisask, and recent results of Sanders concerning Roth’s theorem and Freiman’s theorem. (I almost certainly won’t be able to cover all of those topics, even leaving out details, but I’ll see what I can do.)

The style of the course is slightly experimental, but if it succeeds then it will provide a useful overview of a range of techniques that you should know about if you are contemplating (or actually doing) research in additive or extremal combinatorics.

Ramsey Theory (M16)

I.B.Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden’s theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of ‘classical’ Ramsey theorems, such as Gallai’s theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems.

We hope to cover the following material.

Monochromatic Systems

Ramsey’s theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden’s theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai’s theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado’s theorem. Applications. (m, p, c) -sets and Deuber’s theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman’s theorem.

Infinite Ramsey Theory

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey’s theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Appropriate books

B. Bollobás, *Combinatorics*, C.U.P. 1986

R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley 1990

Combinatorics (M16)

A. Thomason

Combinatorics is used here in its traditional specific sense of combinatorial set theory, involving the study of families \mathcal{A} of subsets of some finite set. What can be said about \mathcal{A} if its members satisfy some special property, often specified in terms of size, intersection or containment? It is the nature of the subject that many natural and innocuous-sounding questions are difficult to address, and it is not possible to get to grips with them in a course of this kind. Nevertheless, the basic approaches that we introduce are already enough to make suprising progress.

The following basic topics will be considered. *Antichains*. The inequalities of Sperner and LYM. *Shadows*. How many subsets must be contained in members of \mathcal{A} ? The Kruskal-Katona theorem shows the way. *Intersecting systems*. The Erdős-Ko-Rado theorem is the first piece of information about \mathcal{A} if every pair of members intersect. *Exact intersections*. This refers to properties of the kind $|A \cap B| \equiv t \pmod{m}$. Theorems of Fisher and Frankl-Wilson have striking applications.

Some of these further topics will also receive an audience. *Isoperimetric inequalities*. On the cube and elsewhere. *Shattering and projections*. What do we see when a set family is projected onto a subset of its groundset? *Orthogonality*. The Lovász Theta function and its uses.

Pre-requisite Mathematics

This is a first course in combinatorics and as such is self-contained, though a familiarity with combinatorial arguments and fundamental theorems such as are found in the Part II Graph Theory course will be helpful.

Level

Literature

1. Bollobás, B. *Combinatorics*, CUP (1986).

Geometry and Topology

Spectral Geometry (L24)

D. Barden

The aim of this course is to give an overview of the work that has blossomed in response to Mark Kac's naive sounding question, posed in 1966: 'Can one hear the shape of a drum?' In other, more general, words can one determine the geometry of a Riemannian manifold from the spectrum, the set of eigenvalues together with their multiplicities, of the Laplacian operator. The answer is

unsurprisingly, no: many pairs, and even continuous families of manifolds, have since been constructed that are isospectral (have the same spectrum) yet are not isometric. BUT

surprisingly, almost yes: these examples are very special, usually highly symmetric, so that it is still possible that generically (a word that may be defined to suit the context) manifolds are spectrally determined. In fact this has already been shown to be the case in certain contexts.

Contents

Chap 1: Definitions and basic results.

Chap 2: Computation of Spectra: flat tori and round spheres. Spectral determination of 2-dimensional flat tori. Examples of isospectral but non-isometric pairs of 4-tori and of planar domains with Dirichlet boundary conditions.

Chap 3: The Heat kernel and some spectrally determined geometric properties.

Chap 4: Sunada's Theorem. The trace formula and the residuality of 'bumpy' metrics.

Chap 5: Riemannian surfaces. The use of Sunada's techniques to construct isospectral non-isometric pairs of Riemann surfaces, with particular attention to those of low genus.

Chap 6: Wolpert's Theorem that generic Riemann surfaces are spectrally determined.

Chap 7: More general examples of isospectral non-isometric manifolds.

Lecture notes will be provided that cover all the above. The course itself will select which topics to emphasise to include some more recent results and to follow the audience's preferences.

Pre-requisite mathematics

Basic Riemannian geometry (as in the Part III Michaelmas course), analysis (including functional analysis) and topology (Part II level) as well as group theory (Part I level). The subject is very much interdisciplinary. However results that are needed from analysis and algebra will be stated mostly without proof, so the level of knowledge required will be that which is sufficient to understand and apply the statements of the theorems rather than knowing or understanding their proofs.

Introductory reading

Apart from brushing up on the pre-requisites, there is nothing that is truly apposite: the best may be the LMS Student Text no. 31 by S. Rosenberg entitled 'The Laplacian on a Riemannian Manifold'. The book 'Eigenvalues in Riemannian Geometry' by Isaac Chavel is introductory in the sense of showing the historic basis (c.1980) from which the more recent results have flowed. However, being published in 1984, it does not cover those developments and does contain a lot more detailed analysis than it is intended to include in the course. The 'Survey of Isospectral Manifolds' by Carolyn Gordon published in Vol. I of the

Handbook of Differential Geometry (published by Elsevier Science in 2000) gives an excellent overview of the subject and the place within it of the topics chosen for the course.

Supporting materials

Complete lecture notes and three example sheets will be made available. For further reading the Gordon survey contains an extensive list of references and a further one will be handed out during the course. The course itself should provide some guidance to these.

Algebraic Geometry (M24)

C. Birkar

This course is an introduction to basic concepts and techniques of algebraic geometry. Algebraic geometry is closely related to number theory, analytic and differential geometry, topology, algebra, logic, physics, etc. Roughly speaking, it is the study of common zeros of a finite collection of polynomials, called an algebraic set. As the name suggests, this study is carried out using algebraic machinery and geometric insight.

If time permits the following topics will be discussed: affine and quasi-projective varieties, regular and rational maps, dimension, singularities, normal varieties, projections, divisors, differential forms, Riemann-Roch theorem for curves, algebraic groups, quotients by group actions, sheaves, and cohomology.

Pre-requisite Mathematics

Familiarity with algebra (rings, modules, fields) is essential for this course. Students are encouraged to take the Part III commutative algebra course.

Literature

1. I.R. Shafarevich. *Basic algebraic geometry I*. Springer 1994.
2. J. Harris. *Algebraic geometry: a first course*. Springer 1992.
3. R. Hartshorne. *Algebraic geometry*. Springer 1977.

Riemannian Geometry (L24)

Alexei Kovalev

This course is a possible natural sequel of the course Differential Geometry offered in Michaelmas Term. We shall explore various techniques and results revealing intricate and subtle relations between Riemannian metrics, curvature and topology. I hope to cover much of the following:

A closer look at geodesics. Brief review from the Differential Geometry course. Jacobi fields, completeness and the Hopf–Rinow theorem. Variations of energy, Bonnet–Myers diameter theorem and Synge’s theorem.

Hodge theory and Riemannian holonomy. The Hodge star and Laplace–Beltrami operator. The Hodge decomposition theorem. Bochner–Weitzenböck formulae. Holonomy groups. Holonomy and curvature. Symmetric spaces. Holonomy and the de Rham cohomology.

Ricci curvature. Fundamental groups and Ricci curvature. Gromov’s proof of Gallot’s bound on the first Betti number. The Cheeger–Gromoll splitting theorem.

The lectures will be supplemented by three example classes.

Pre-requisite Mathematics

Manifolds, differential forms, vector fields. Basic concepts of Riemannian geometry (curvature, geodesics etc.) and Lie groups. The course *Differential Geometry* offered in Michaelmas Term is the ideal pre-requisite.

Literature

1. S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry*. Springer-Verlag, 1990.
2. M.P. do Carmo, *Riemannian geometry*. Birkhäuser, 1993.
3. I. Chavel, *Riemannian geometry, a modern introduction*, CUP 1995.
4. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983. Chapter 6.
5. P. Petersen, *Riemannian geometry*, Springer-Verlag, 1998.
6. A.L. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

The first few chapters of do Carmo's text provide a good introductory reading.

Symplectic Geometry (M16)

Gabriel P. Paternain

Symplectic geometry is the geometry that governs classical physics and this course will provide a basic introduction to the subject following the text by Ana Cannas de Silva *Lectures on Symplectic Geometry*. It will be useful for various other Part III courses in the Lent term (e.g. *Hamiltonian Dynamics* and *Homological Mirror Symmetry*).

We will try to cover at least the first four parts in Anna Cannas da Silva's text.

Pre-requisite Mathematics

The *Differential geometry* course offered in the Michaelmas term is essential. The *Algebraic Topology* course also offered in the Michaelmas term will be very useful. All the necessary background on manifolds and differential forms can be found in references [4] and [5] below.

Literature

1. Ana Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001.
2. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.
3. D. McDuff; D. Salamon, *Introduction to Symplectic Topology*, Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.
4. V. Guillemin, A. Pollack, *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.
5. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, 94. Springer-Verlag, New York-Berlin, 1983.

Knots and Four-manifolds (L24)

J. Rasmussen

The study of four-dimensional manifolds occupies a unique position in geometric topology. Four-manifolds are complicated enough to be interesting in ways that low-dimensional manifolds are not. For example, there are four-manifolds which are homeomorphic, but not diffeomorphic. Such behaviour occurs for higher dimensional manifolds as well, but the extra “wobble room” available in high dimensions gives us reasonably good control over these extra smooth structures. Thus there are many interesting phenomena which occur only in dimension four, such as the presence of infinitely many different smooth structures on a single topological manifold, or the existence of an “exotic” \mathbf{R}^4 (*i.e.* a manifold homeomorphic but not diffeomorphic to \mathbf{R}^4).

This course will be an introduction to four-dimensional topology as seen through the lens of knot theory. The main goal will be give a knot-theoretic proof of the existence of an exotic \mathbf{R}^4 . Along the way, we will touch on some important invariants of knots (the Alexander and Jones polynomials), and of four-manifolds (Casson-Gordon invariants, minimal genus of embedded surfaces.)

I intend to cover the following topics:

Handle decompositions of manifolds. Dehn surgery, Kirby calculus.

Alexander polynomial. Knot genus, Fox calculus.

Branched double covers. Rational knots, signature and determinant, slice genus, Casson-Gordon invariants.

Jones polynomial. Tait conjectures, Khovanov homology.

Topologically slice knots. Freedman’s slice theorem, bounds on the smooth slice genus, existence of exotic \mathbf{R}^4 .

Pre-requisite mathematics: Algebraic Topology. No previous knowledge of geometric topology is assumed.

References:

1. L. Kauffman, *On Knots*, PUP, 1987.
2. R. Gompf and A. Stipsicz, *4-manifolds and Kirby Calculus*, AMS, 1999.
3. D. Rolfsen, *Knots and Links*, AMS, 1976.

Differential Geometry (M24)

J. Ross

This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, smooth maps. Tangent vectors and vector fields, tangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Lie Groups. Differential 1-forms, cotangent bundle.
- *Vector Bundles.* Structure group. The example of Hopf bundle. Bundle morphisms and automorphisms. Exterior algebra of differential forms. Tensors. Orientability of manifolds. Partitions of unity and integration on manifolds, Stokes Theorem; de Rham cohomology. Lie derivative of tensors. Connections on vector bundles and covariant derivatives: covariant exterior derivative, curvature. Bianchi identity, orthogonal connections.

- *Riemannian Geometry*. Connections on the tangent bundle, torsion. Bianchi's identities for torsion free connections. Riemannian metrics, Levi-Civita connection, Christoffel symbols, geodesics. Riemannian curvature tensor and its symmetries, second Bianchi identity, sectional curvatures. Ricci tensor and Einstein metrics. Ricci and scalar curvatures. Schur's theorem.

The main references for this course are the books listed below and some printed notes on the lecturer's home page.

References

1. D. Barden, C. Thomas, *An introduction to differentiable manifolds*. Imperial College Press, 2003.
2. R.W.R. Darling, *Differential forms and connections*. CUP, 1994.
3. M. Spivak, *Differential Geometry, Volume 2*. Publish or Perish, 1999.
4. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983.

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus.

Abelian Varieties (L24)

M. Shen

Abelian varieties are very special algebraic varieties that are interesting in both algebraic geometry and number theory.

In this course, we will develop the basic theory of abelian varieties. We first introduce the analytic theory of complex tori and get a description of line bundles on those objects. After getting the picture over the field of complex numbers, we move on to develop the theory over an arbitrary field using the language of schemes. This can be viewed as a nice application of the techniques one learned from an algebraic geometry course. This part includes the construction of dual abelian variety and the computation of cohomology of line bundles. After setting up the basic theory, we will discuss some topics of more arithmetic flavor. This might include a study of the endomorphism algebra and its action on Tate module; positivity of Rosati involution; Mordell-Weil theorem. If time allows, we will give some examples of abelian varieties that naturally arise in geometry. This includes Jacobian of curves, Albanese varieties, Picard varieties and intermediate Jacobians. For most part of the course, we will follow Mumford's book below.

List of topics: Complex tori, Theta functions, theorem of the cube, dual abelian varieties, cohomology of line bundles; endomorphism algebra, Tate module, Rosati involution, Mordell-Weil theorem, Jacobian, Picard variety.

Pre-requisite Mathematics

Part III algebraic geometry is required; A course on complex geometry would be helpful but not required.

Literature

1. D. Mumford, *Abelian Varieties*.
2. R. Hartshorne, *Algebraic Geometry*. Springer (1977).

Algebraic Topology (M24)

N. I. Shepherd-Barron

Algebraic topology (AT) is the construction of algebraic objects out of geometric ones in order to prove theorems and solve problems in geometry and algebra. More specifically we'll construct various forms of homology and prove them to be the same. Then we'll prove Poincaré duality. After that we will show how Morse theory can be used to calculate and to prove theorems (weak Lefschetz). Then it will be time to introduce cohomology and prove de Rham's theorem. More generally, we'll discuss the cohomology of sheaves. One point we'll see is that cohomology is in fact more flexible than homology.

Of course there will be many examples along the way. Most will come from algebraic geometry because (a) I am an algebraic geometer; (b) examples have to come from somewhere; (c) the relationship between AT and AG enriches both subjects; (d) AT is, like anything worthwhile, a large and varied subject and no course can begin to teach more than some limited aspects of it.

Homological mirror symmetry (L24)

Non-Examinable (Graduate Level)

Ivan Smith

Kontsevich's Homological Mirror Symmetry conjecture is a large and ambitious program relating various aspects of algebraic and symplectic geometry through their homological invariants (Ext groups and categories of coherent sheaves on the one hand, Floer cohomology and categories of Lagrangian submanifolds on the other). Whilst many aspects of mirror symmetry remain conjectural, the basic objects can be defined and compared in a number of interesting and non-trivial cases. For reasons of time and the lecturer's (lack of) expertise, this course will be a "topics course" rather than a systematic development of the subject. Topics to be covered at varying levels of detail may include: Floer cohomology and the Fukaya category, derived categories of sheaves, mirror symmetry for the elliptic curve and the four-dimensional torus, superpotentials and Lagrangian torus fibrations, applications to symplectic topology.

Desirable Previous Knowledge

Assumed background will vary considerably from place to place, but reasonable exposure to all of algebraic geometry, symplectic geometry and homological algebra will be essential.

Introductory Reading

1. *Calabi-Yau manifolds and related geometries*. Mark Gross, Daniel Huybrechts and Dominic Joyce. Springer-Verlag, 2003.

Comment: The chapter by Gross is most relevant, followed by that of Joyce.

Reading to complement course material

1. *Homological algebra of mirror symmetry*. Maxim Kontsevich. Proceedings of the International Congress of Mathematicians, Zurich 1994.

Comment: where it all started, and still astonishing.

2. *Fukaya categories and Picard-Lefschetz theory*. Paul Seidel. EMS (2008).

Comment: Demanding but essentially the only reference for much of the material it covers.

Hamiltonian Dynamics (L24)

A. Sorrentino

This course is an introduction to the theory of Hamiltonian systems of differential equations. These systems arise and play a fundamental role in many different contexts, ranging from pure mathematics (dynamical systems, geometry, symplectic topology, *etc.*) to more applied subjects (physics, astronomy, chemistry, *etc.*).

What is a Hamiltonian system? Roughly speaking, it is a system of first order ordinary differential equations, which are expressed in a simple and symmetric form in terms of a function, called the *Hamiltonian*. This way of reformulating Newton's equations can be traced back at least to Sir William Rowan Hamilton ("On the application to dynamics of a general mathematical method previously applied to optics", 1834), although equations of similar form had already been considered by Lagrange and Poisson several years before. However, Hamiltonian dynamics is much more than just a "reformulation"; it led, for example, to Poincaré's geometrical insight that gave rise to what nowadays is called symplectic geometry and to many other very active fields of research.

In studying the dynamics of a system, the main (utopian) aim would be to characterize completely the totality of motions and their qualitative properties. Given an initial state, what will the future "evolution" of the system be? What was its past "development"?

The deterministic nature of these systems clashes with the difficulty in trying to answer these questions. In fact, except for few special, but interesting, cases for which everything is clearly understood (the so-called *integrable case*), in general the nature of these systems seems to be very complex and varied, with a subtle - and not easy to understand - balance between "regular" motions and "chaotic" ones.

The main goal of this course is to describe different approaches and techniques that have been developed in order to answer the above questions. In particular, we shall cover (some of) the following topics:

- Symplectic geometry and Hamiltonian vector fields;
- Classical theory: canonical formalism and generating functions;
- Symmetries and Integrability;
- Perturbation theory and KAM theory;
- Stability vs. Instability: Nekhoroshev theory and Arnold's diffusion.
- The principle of least action: Aubry-Mather theory;
- Hamilton-Jacobi equation and weak KAM theory;
- Twist maps and billiards;
- Applications to celestial mechanics.

Prerequisite Mathematics

There are no formal prerequisites, although the *Symplectic geometry* course offered in the Michaelmas term could be very useful.

Literature

1. V. I. Arnol'd, *Mathematical methods of classical mechanics*, Springer-Verlag, 1989.
2. V. I. Arnol'd, V.V. Kozlov and A. I. Neishtadt, *Dynamical systems III*, Encyclopaedia of Mathematical Sciences, Springer-Verlag, 1988.
3. A. Cannas da Silva, *Lectures on symplectic geometry*, Springer-Verlag, 2001.

4. R. de la Llave, *A tutorial on KAM theory*, AMS, 2001.
5. C. Golé, *Symplectic twist maps. Global variational techniques*, World Scientific Publishing Co., 2001.
6. H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, 1995.
7. R. S. MacKay and J. D. Meiss, *Hamiltonian dynamical systems*, Adam Hilger, 1987.
8. K. R. Meyer and G. R. Hall, *Introduction to Hamiltonian dynamical systems and the N-body problem*, Springer-Verlag, 1992.
9. K. F. Siburg, *The principle of least action in geometry and dynamics*, Springer-Verlag, 2004.

Galois Cohomology (L24)

C. Vial

The cohomology of Galois groups of fields is an essential tool in modern algebraic geometry and number theory. This course aims at introducing the basics of this theory.

I will start by defining central simple algebras over a field and prove several of their properties. This will lead to the notion of Brauer group of a field. Then using Galois descent I will show how this Brauer group can be identified with a certain first cohomology group. I will also briefly discuss the case of quadratic forms and Severi-Brauer varieties.

We will then introduce the cohomology of finite groups with values in abelian modules. In particular, we will interpret these cohomology groups as being classes of group extensions. We will also compute the cohomology of finite cyclic groups.

The Galois group G of a field k is not finite in general. Rather it is a profinite group which is endowed with a natural topology. The Galois cohomology of k with values in a certain module is defined to be the group cohomology of G with values in a certain abelian continuous G -module. We will give a cohomological interpretation of the Brauer group of k in the context of Galois cohomology.

We will then be interested in the computation of the Galois cohomology of several types of fields : finite fields, local fields and global fields (in particular global fields of positive characteristic). For this purpose we will introduce the notion of cohomological dimension and prove that several fields are C_1 . Time permitting, we will discuss some possible applications to local class field theory and the Hasse principle, to the weak Mordell-Weil theorem and to some duality theorems.

Finally we will introduce Milnor's K-theory of fields, do some computations on several examples and construct the Galois symbol. We will state the Bloch-Kato conjecture, as well as the Merkurjev-Suslin theorem and show some of its consequences on central simple algebras. In another direction, we might also discuss cohomological invariants of algebraic groups and work these out in the case of finite cyclic groups and quadratic forms.

Pre-requisite Mathematics

Some familiarity with Galois theory, commutative algebra and algebraic geometry would be helpful.

Literature

1. Serre. Galois cohomology. Springer.
2. Serre. Local fields. Springer.
3. Gille & Szamuely. Central simple algebras and Galois cohomology. CUP.

Complex Manifolds (L24)

P. M. H. Wilson

The course builds on Differential Geometry. The following is an outline of the syllabus, which will be subject to minor changes

- Fundamentals of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles. Canonical line bundles, normal bundle for a submanifold and the adjunction formula. Examples including projective space, hypersurfaces and blowups.
- Sheaf cohomology; de-Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles, in particular holomorphic bundles.
- Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be presented without proof).
- Compact Kähler manifolds. The Kähler Identities. Ricci Curvature. Examples, such as the Fubini-Study metric on projective space.
- Hodge and Lefschetz decompositions on cohomology, Kodaira-Nakano vanishing.
- The Kodaira embedding theorem.
- Uniformization for curves and Special Kähler metrics in higher dimensions (if time allows)

Prerequisites

Part III Differential Geometry

Some familiarity with the analysis of one complex variable.

References

1. D. Huybrechts, *Complex Geometry*, Springer (2005).
2. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).
3. R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer (1980).
4. G. Tian, *Canonical Metrics in Kähler Geometry*, Birkhäuser (2000).
5. S. Kobayashi and K. Nomizu. *Foundations of differential geometry*, vol 2. Wiley (1996).

Topos Theory (L24)

Dr. Olivia Caramello

The class of categories known as toposes were first introduced by Alexander Grothendieck in the early 1960s, in order to provide a mathematical underpinning for the ‘exotic’ cohomology theories needed in algebraic geometry. Every topological space X gives rise to a topos (the category of sheaves of sets on X) and every topos in Grothendieck’s sense can be considered as a ‘generalized space’.

At the end of the same decade, William Lawvere and Myles Tierney realized that the concept of Grothendieck topos also yielded an abstract notion of mathematical universe within which one could carry out most familiar set-theoretic constructions, but which also, thanks to the inherent ‘flexibility’ of the notion of topos, could be profitably exploited to construct ‘new mathematical worlds’ having particular properties.

A few years later, the theory of classifying toposes added a further fundamental viewpoint to the above-mentioned ones: a topos can be seen not only as a generalized space or as a mathematical universe, but also as a suitable kind of first-order theory (considered up to a general notion of equivalence of theories).

Recently, a new view of Grothendieck toposes as unifying spaces in Mathematics being able to serve as ‘bridges’ for transferring information between distinct mathematical theories has emerged. This approach, first introduced in the lecturer’s Ph.D. thesis, has already generated ramifications into distinct areas of Mathematics and points towards a realization of Topos Theory as a unifying theory of Mathematics.

The course will begin by presenting the basic theory of toposes and geometric morphisms, with the aim of reaching these last developments. By the end of the term, the student will have acquired tools and methods to study mathematical theories from a topos-theoretic perspective, extract new information about mathematical dualities, and establish new and fruitful connections between distinct fields.

The course will be backed up by four example classes.

Desirable Previous Knowledge

Knowledge of the material of the Michaelmas Term course on Category Theory is essential. Some familiarity with classical first-order logic (such as is provided by the Part II course on Logic and Set Theory) would be very desirable but not essential. No previous knowledge of sheaf theory is required.

Appropriate literature

1. McLarty, C.: *Elementary Categories, Elementary Toposes*, Oxford U.P. 2002. Recommended for preliminary reading; it’s a very gently-paced introduction to the subject, written for philosophers with little mathematical background.
2. MacLane, S. and Moerdijk, I.: *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*, Springer-Verlag, 1992. The best available textbook on the subject, though its approach diverges in some respects from that which will be adopted in the course.
3. Johnstone, P.T.: *Sketches of an Elephant: a Topos Theory Compendium*, Oxford U.P. (2 volumes), 2002. Emphatically not a textbook, but the best place to find out anything you want to know about toposes.
4. Caramello, O.: *The Duality between Grothendieck Toposes and Geometric Theories*, Ph.D. thesis, University of Cambridge, 2009. The place to learn about the view of classifying toposes as unifying spaces and its technical implications.
5. Caramello O.: *The Unification of Mathematics via Topos Theory*, arXiv:math.CT/1006.3930v1, 2010. The programmatic paper on the view ‘toposes as bridges’.

Computable Function Theory (L16)

T. Forster

Structural induction and wellfounded induction. Primitive recursive functions, General recursive functions. Ackermann's function. Goodstein's function. Turing Machines. Decidable and semidecidable sets. Existence of a universal Turing machine. Kleene's T -function. Unsolvability of the Halting Problem. Fixed-point theorem. Rice's theorem. Hierarchies of fast-growing functions. Trakhtenbrot's theorem. Productive sets. Gödel's β -function. Gödel's incompleteness theorem. Matijasiević's theorem on diophantine representation. Recursive ordinals. Representation of computable functions by λ -terms. Kleene-Post. Friedberg-Muchnik and the Priority method. Baker-Gill-Solovay.

Pre-requisite Mathematics

I shall assume that everybody has done Part II Set Theory and Logic, though this is actually overkill.

Literature

There are numerous good books on this subject. The following is in paperback, and holders of a Cambridge Blue card can acquire it from the CUP bookshop in town for a 15% discount.

1. Cutland, N. Computability, Cambridge University Press

Quine Systems (M8)

Non-Examinable (Graduate Level)

T. Forster

Quine's 1937 article 'New Foundations for Mathematical Logic' unleashed on the mathematical world an axiomatic set theory with some bizarre and interesting features. It is by far the most interesting and the most problematic of the alternatives to ZF (which it probably subsumes, in any case). Its consistency is still an open question, perhaps the oldest open problem in Set Theory. This course will explain the history, discuss the motivation, and present as much of the technical development as can be done in 8 lectures. The material is too esoteric for Part III, but the course will be at a Part III level of difficulty. There is no recommended reading, (tho' the lecturer's OUP monograph 'Set Theory with a Universal Set' is an obvious choice for those who want to bone up in advance). The pace and content will be determined in real time by audience reaction and there will be a prize for the best set of notes taken by an attendee.

Category Theory (M24)

Dr Julia Goedecke

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics are a rough guide to what will be covered in the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories.

Locally small categories. The Yoneda lemma. Representations of functors.

Limits. Construction of limits from products and equalizers. Preservation and creation of limits.

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. The Adjoint Functor Theorems.

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions.

Abelian categories. Exact sequences, projective resolutions. Introduction to homological algebra.

Pre-requisite Mathematics

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. Mac Lane, S. *Categories for the Working Mathematician*, Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.
2. Awodey, S. *Category Theory*, Oxford U.P. 2006. A new treatment very much in the spirit of Mac Lane’s classic, but rather more gently paced.
3. Borceux, F. *Handbook of Categorical Algebra*, Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.
4. McLarty, C. *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P. 1992. A very gently-paced introduction to categorical ideas, written by a philosopher for those with little mathematical background.

Two-dimensional category theory. (M24)

Non-Examinable (Graduate Level)

Ignacio López Franco

This course is aimed to provide some of the fundamental theory of categories with extra structure. The course is divided in two parts. In the first we study monoidal (or tensor) categories and related notions, as for example braided categories. These types of categories gained importance for their close relationship with quantum groups (Hopf algebras) and the Yang-Baxter equation and now are ubiquitous in many areas of mathematics. The second part covers 2-categories and the basics of two-dimensional monad theory. This allows us to study categories with extra structure –*e.g.*, monoidal categories– and functors weakly preserving this structure –*e.g.*, monoidal functors– in a systematic way. The course ends by looking to general coherence results.

Desirable Previous Knowledge

Some knowledge of category theory is desirable, as provided by the Part III Category Theory course.

Introductory Reading

1. S. Lack, *A 2-categories companion*. Baez, John C. (ed.) et al., Towards higher categories. Berlin: Springer. The IMA Volumes in Mathematics and its Applications 152, 105-191 (2010).
2. C. Kassel, *Quantum groups*. Graduate Texts in Mathematics 115, Springer-Verlag, 1995.

Reading to complement course material

1. R. Blackwell, G. M. Kelly and A. J. Power, *Two-dimensional monad theory*, J. Pure Appl. Algebra **59** (1989), 1–41.
2. A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. **102** (1993), 20–78.
3. S. Lack, *Codescent objects and coherence*, J. Pure Appl. Algebra **175** (2002), 223–241.
4. G. M. Kelly and R. Street, *Review of the elements of 2-categories*, Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 75-103. Springer, 1974.
5. G. M. Kelly, *Elementary observations on 2-categorical limits*, Bull. Aust. Math. Soc. **39** (1989), 301–317.

Decision Methods over Real and Algebraically Closed Fields. (L16)

Non-Examinable (Graduate Level)

Grant Olney Passmore

An axiomatic theory T is called *decidable* if there exists an algorithm which, for any conjecture φ in the language of T , can determine in finitely many steps whether or not φ is a theorem of T . Many core axiomatic theories, including the elementary arithmetical theories of \mathbb{Z} and \mathbb{Q} , are not decidable. Amazingly, the elementary theories of algebra and geometry over \mathbb{R} and \mathbb{C} are decidable. This ten lecture course will study decision methods for elementary algebra and geometry over \mathbb{R} and \mathbb{C} . We will examine methods based on Gröbner bases (\mathbb{C}), Muchnik sign matrices (\mathbb{C} and \mathbb{R}), cylindrical algebraic decomposition (\mathbb{R}) and Stengle's Weak Positivstellensatz (\mathbb{R}). The machinery developed should be of interest to logicians, algebraists and algebraic geometers.

Desirable Previous Knowledge

A basic grounding in first-order logic, field theory and general topology. (By 'basic grounding' I mean that good undergraduate courses in these areas should be more than sufficient.)

Introductory Reading

1. Chapter 1 of Adams, W.W. and Loustaunau, P., An Introduction to Gröbner Bases, AMS Grad. Studies in Maths., 1994.
2. Appendix A.5 of Hodges, W. Model Theory. Cambridge Univ. Pr., 1997.
3. Chapter 1 of Sturmfels, B. Gröbner Bases and Convex Polytopes, AMS Univ. Lecture Series, 1995.
4. Sturmfels, B. What is a Gröbner Basis? Notices of the AMS (Vol. 52; Num 10).

Reading to complement course material

I will give handouts covering all of the material we work through in class. In addition, the following resources may be helpful:

1. Caviness, B.F. and Johnson, J.R. (eds), Quantifier Elimination and Cylindrical Algebraic Decomposition, Springer, 1998.
2. Basu, S., Pollack, R. and Roy, M-F. Algorithms in Real Algebraic Geometry (2nd ed.). Springer, 2006.
3. Passmore, G.O., Combined Decision Procedures for Nonlinear Arithmetics, Real and Complex, PhD thesis, University of Edinburgh, 2011.

Number Theory

Introduction to Iwasawa theory (M24)

John Coates

Iwasawa theory is a p -adic method for studying the mysterious connexion between special values of complex L -functions and the basic problems of arithmetic geometry, especially the exact formulae such as those found in the conjecture of Birch and Swinnerton-Dyer. The course will be an introduction to the subject, assuming no prior knowledge of Iwasawa theory. However, it will assume knowledge of basic algebraic number theory. In addition, it will carefully recall the statements of the basic results of local and global class field theory, and then make systematic use of these. After some preliminary material, the course will concentrate on the simplest class of arithmetic Iwasawa modules, and mainly discuss work in Iwasawa's fundamental paper "On \mathbf{Z}_1 -extensions of algebraic number fields", *Ann. of Math.* 98 (1973), 246-326.

References

1. J. Coates, R. Sujatha, *Cyclotomic fields and zeta values*, Springer 2006.
2. L. Washington, *Introduction to cyclotomic fields*, Springer 1982.
3. K. Iwasawa, *On \mathbf{Z}_1 -extension of algebraic number fields*, *Ann. of Math.* 98 (1973), 246-326.

Topics in Iwasawa Theory (L24)

Non-Examinable (Graduate Level)

John Coates

This is a continuation of the Part III course dealing with more advanced topics.

Elliptic Curves (M16)

Vladimir Dokchitser

This is an introductory course on the arithmetic of elliptic curves. The main focus will be on the group of rational points of elliptic curves over \mathbf{Q} . The course will mainly concern the algebraic aspects of the theory, although we will also make use of p -adic methods and complex analysis. The topics I plan to cover are:

Elliptic curves. The group law, Weierstrass form, j -invariant.

Elliptic curves over \mathbf{C} . Lattices in \mathbf{C} , doubly periodic functions, Weierstrass parametrisation.

Elliptic curves over \mathbf{Q}_p . Minimal models, reduction modulo p , the group of points over finite fields, the formal group of an elliptic curve.

Elliptic curves over \mathbf{Q} . Statement of the Mordell-Weil theorem, determination of the torsion subgroup, descent via a 2-isogeny; heights, the weak Mordell-Weil theorem, proof of the Mordell-Weil theorem.

If time allows: Hasse-Weil Theorem, Tate module, Weil pairing, Selmer and Tate-Shafarevich groups.

There will be example sheets and examples classes.

Pre-requisite mathematics

Some familiarity with Galois theory, p -adic numbers, complex analysis and number fields is desirable.

Literature

1. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.
2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Local Fields (M24)

Dr T A Fisher

The theory of local fields was introduced by Hensel in the early 1900's as an alternative approach to algebraic number theory. The basic idea is to consider the completions of a number field K at all absolute values, not just the ones arising from the embeddings of K into the reals or complexes. One can then borrow techniques from analysis to study K and its finite extensions in a way that focuses on their behaviour at just one prime. For instance the analogue of the Newton-Raphson method for root finding goes by the name of Hensel's lemma.

Nowadays, local fields have established themselves as a natural tool in many areas of number theory and also in subjects like representation theory, algebraic topology and arithmetic geometry (e.g. elliptic curves). So this course is likely to be useful for those taking the "Class Field Theory" and "Elliptic Curves" courses.

The course will begin by introducing the field of p -adic numbers \mathbb{Q}_p (where p is a prime). This is the completion of the field of rational numbers \mathbb{Q} with respect to the p -adic absolute value defined for non-zero $x \in \mathbb{Q}$ by $|x|_p = 1/p^n$ where $x = p^n a/b$ with p not dividing a or b . Topics to be covered will then include: absolute values on fields, valuations, complete fields and their extensions, the different and discriminant, decomposition groups, inverse limits, Hensel's lemma and ramification theory. If time permits, then possible further topics include: Skolem's method, local class field theory (statements only), the Hilbert norm residue symbol, and the Hasse-Minkowski theorem.

Pre-requisite Mathematics

Basic algebra up to and including Galois theory is essential. It will be assumed students have been to a first course in algebraic number fields.

Literature

1. J.W.S. Cassels, *Local fields*, CUP, 1986.
2. G.J. Janusz, *Algebraic number fields*, AMS, 1996.
3. N. Koblitz, *p -adic numbers, p -adic analysis and zeta-functions*, Springer, 1977.
4. J. Neukirch, *Algebraic number theory*, Springer, 1999.
5. J.-P. Serre, *A course in arithmetic*, Springer, 1973.
6. J.-P. Serre, *Local fields*, Springer, 1979.

Prime Numbers (L24)

B. J. Green

This course will give an introduction to analytic number theory via the study of prime numbers. Topics may include

- Basic theory of the Riemann ζ -function and proof of the prime number theorem. Outline of the theory of Dirichlet L -functions, Siegel zeros and primes in arithmetic progressions.
- The Selberg sieve and applications such as upper bounds for twin primes and the work of Goldston-Pintz-Yıldırım on small gaps between primes. Discussion of the parity problem and Bombieri's asymptotic sieve.
- Vinogradov's theorem that every large odd number is the sum of 3 primes.
- Further topics as time allows, for example the Erdős-Selberg "elementary" proof of the prime number theorem, the proof that the primes contain arbitrarily long arithmetic progressions, or random matrix theory.

References and Prerequisites

I plan to produce notes for the course. An excellent text for most of the topics is Davenport's *Multiplicative Number Theory*, Springer GTM 74, but my treatment will often diverge from his.

The only formal prerequisite is a working knowledge of complex analysis.

p-adic Hodge Theory (M24) *Non-Examinable (Graduate Level)*

W. Kim

Initially, p -adic Hodge theory began its life as an attempt to find a good analogue of de Rham theorem for p -adic cohomologies (such as étale, de Rham, and crystalline cohomologies). In 1970's Fontaine's constructed "big rings" where "periods" of cohomology classes should lie (so-called "period rings") and formulated (then conjectural) analogue of de Rham theorem. The construction of period rings also provides a hierarchy of p -adic Galois representations of a p -adic field and a means to study certain nice classes of p -adic representations via associating suitable semilinear algebra objects to them, and this is important in its own right (for example, in relation to the study of Galois representations coming from modular forms/automorphic representations).

During the past 20-30 years, there have been incredible achievements in p -adic Hodge theory; for example, the proof of comparison theorems for p -adic cohomologies (due to Faltings, Tsuji, et. al. to name just a few), the proof of foundational conjectures (such as weak admissibility implying admissibility and p -adic monodromy conjecture), and some new approaches (such as (ϕ, Γ) -modules, relationship with p -adic local Langlands correspondence for $GL_2(\mathbf{Q}_p)$, integral p -adic Hodge theory).

In this course we give a gentle introduction/overview of p -adic Hodge theory, aiming for the understanding of basic objects and the statements of fundamental results in p -adic Hodge theory. We will review suitable backgrounds and may skip some highly technical proofs. We will cover some advanced topics if time permits.

- Motivations: Galois representations (ℓ -adic and p -adic), Tate modules (for p -divisible groups), Hodge-Tate decomposition for Tate modules, statements of p -adic comparison theorems, etc.
- Period rings (B_{dR} , B_{cris} , B_{st} , etc.)
- "Period rings formalism", de Rham/semi-stable/crystalline representations
- Filtred (ϕ, N) -modules, admissibility and weak admissibility
- Potentially semi-stable representations and Weil-Deligne representations, p -adic monodromy theorem
- (optional) Breuil's theory of strongly divisible modules, review of p -divisible groups and Dieudonné crystals, "crystalline comparison isomorphism" for p -divisible groups
- (optional) (ϕ, Γ) -modules and p -adic differential equations (a la Sen, Fontaine, Colmez, Berger)

Pre-requisite Mathematics

- Algebraic number theory (for local fields)
- Commutative algebra and some basic algebraic geometry - Knowledge on p -adic cohomologies (such as étale, de Rham, crystalline cohomologies) will *not* be assumed.

Level

Part III (24 lectures; non-examinable).

Literature

1. J.-M. Fontaine, *Exposés II, III, VIII from Périodes p -adiques*, Société Mathématique de France (Astérisque 223), 1994.
2. L. Berger, *An introduction to the theory of p -adic representations from Geometric aspects of Dwork theory, Vol 1*, Walter de Gruyter GmbH & Co. KG, 2004.
3. O. Brinon, B. Conrad, *CMI Summer School Notes on p -adic Hodge Theory (preliminary version*, available at math.stanford.edu/~conrad.
4. C. Breuil, *Integral p -adic Hodge theory from Algebraic geometry 2000, Azumino (Hotaka)*, Mathematics Society of Japan (Adv. Stud. Pure Math. 36), 2002.
5. J. Tate, *p -divisible groups from Local fields (Driebergen, 1966)*, Springer, 1967.

Waring's Problem (E12)

Non-Examinable (Graduate Level)

V.R. Neale

This course is intended as an introduction to one of the central topics of additive number theory, namely Waring's problem and its solution using the Hardy-Littlewood circle method.

The course is tentatively expected to cover the following topics.

Introduction to Waring's problem. The application of the Hardy-Littlewood circle method to the solution of Waring's problem. Weyl's inequality and Hua's lemma. The application of Weyl's inequality to Diophantine approximation. Improved bounds for Waring's problem for higher powers. Applications of the circle method to related problems.

There are a number of references for Waring's problem and the circle method, including the books listed below.

References

1. H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2005.
2. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Colloquium Publications **53**, American Mathematical Society, Providence, Rhode Island, 2004.
3. R.C. Vaughan, *The Hardy-Littlewood Method*, Second edition, Cambridge Tracts in Mathematics **125**, Cambridge University Press, Cambridge, 1997.

It would be helpful to attend Professor Green's course on Prime Number Theory, which will include an application of the circle method to prove Vinogradov's three primes theorem, but no knowledge of the circle method will be assumed.

Rigid analytic geometry and p -adic modular forms (L24)

Non-Examinable (Graduate Level)

J. Newton

This course will cover some topics in the modern theory of modular forms, starting with the algebraic theory of modular curves, and aiming to eventually cover some of the theory of ‘overconvergent’ modular forms. We hope to discuss some of the following areas, depending on the background and interests of the audience.

- Algebraic theory of modular curves and modular forms
- Hecke operators as correspondences on modular curves
- Katz’s theory of mod p and p -adic modular forms
- Rigid analytic geometry
- Overconvergent modular forms
- Relationship between overconvergent and classical modular forms
- Galois representations attached to classical and overconvergent modular forms

Pre-requisite Mathematics

- Local fields
- Elliptic curves
- Helpful to have some previous exposure to algebraic geometry and the classical theory of modular forms

Level

Graduate level (non-examinable).

Literature

1. F. Diamond, J. Shurman *A First Course in Modular Forms*, Springer (Graduate Texts in Mathematics 228), 2005.
2. F. Diamond, J. Im *Modular forms and modular curves*, AMS (Seminar on Fermat’s Last Theorem (Toronto, ON, 1993/1994), CMS Conference Proceedings, 17) 1995.
3. B. Conrad *Arizona winter school lecture on rigid geometry* Sections 1–3, online at <http://math.stanford.edu/~conrad/papers/aws.pdf>.

Topics in Number Theory (L24)

Non-Examinable (Graduate Level)

A J Scholl

This course will be an introduction to the theory of Shimura varieties, particularly those of PEL type, aimed at an audience already acquainted with modular curves. Some prior knowledge of the elementary theory of abelian varieties will certainly be useful; I will not assume any serious knowledge of the general theory of reductive groups, but will review the necessary material as needed.

Literature

1. P. Deligne, “Travaux de Shimura”, Séminaire Bourbaki exposé 389
2. —, “Variétés de Shimura: interprétation modulaire...”, in vol.2 of Corvallis proceedings (AMS PSMP vol.33)

3. J. S. Milne, “Introduction to Shimura varieties”
4. —, “Complex multiplication (preliminary notes)”

[3] and [4] can be found at www.jmilne.org.

Class Field Theory (L24)

T. Yoshida

The class field theory of Takagi-Artin, established in the 1920's, is a landmark in algebraic number theory, incorporating a large part of what was done in the 19th century by Gauss, Dedekind, Kummer, Hilbert and others, and became a starting point of many trends in modern mathematics, including its generalisations by Weil, Tate, Langlands, Deligne and others.

In this course we treat local and global class field theory with modern techniques, and prove them in detail as much as time permits. We will focus on characteristic zero (number fields) case. In the last part of the course, we indicate how it is reformulated into the so-called global Langlands correspondence for GL_1 , and generalized into a framework of Galois representations that relates to motives and automorphic representations (e.g. elliptic curves, modular forms).

- Introduction: quadratic reciprocity, etc.
- Cyclotomic fields (CFT over \mathbf{Q})
- Adeles and ideles, review of global theorems
- Statement of class field theory
- Local Artin map via Lubin-Tate theory
- Abelian categories, cohomological functors
- Group cohomology, Galois cohomology
- Proof of class field theory
- ℓ -adic characters and Hecke characters
- Introduction to global Langlands correspondence

Pre-requisite Mathematics

- Galois theory
- Number fields
- Local fields (we review the basics)

Level

Part III (16 lectures; non-examinable),
Graduate level (8 lectures).

Literature

1. J.-P. Serre, *Local Fields*, Springer (Graduate Texts in Mathematics), 1980.
2. K. Iwasawa, *Local Class Field Theory*, Oxford UP, 1986.
3. E. Artin, J. Tate, *Class Field Theory*, Benjamin, 1967.
4. J. W. S. Cassels, A. Fröhlich, *Algebraic Number Theory*, Academic Press, 1967.
5. A. Weil, *Basic Number Theory*, Springer (Classics in Mathematics), 1995.

Probability

Percolation and Related Topics (M16)

Geoffrey Grimmett

The percolation process is the simplest probabilistic model for a random medium. It has a central role in the general theory of disordered systems arising in the mathematical sciences, and it has strong connections with statistical mechanics. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to the theory of phase transitions in physics.

The basic theory of percolation will be described in this course, with some emphasis on areas for future development. The fundamental techniques, including correlation and/or concentration inequalities and their ramifications, will be covered. The related topics may include self-avoiding walks, and further models from interacting particle systems, and certain physical models for the ferromagnet such as the Ising and Potts models.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following text will cover the majority of the course, and is available freely online.

Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010;
see <http://www.statslab.cam.ac.uk/~grg/books/pgs.html>

Random Matrices (M16)

V. Kargin

This course is about the asymptotics of the eigenvalues of large random matrices, focusing on Wigner-like matrices and the Gaussian Unitary Ensemble. Random matrices find applications in many problems in computer science, physics and statistics. The course will cover the basic methods used in the random matrix theory including the methods of traces, orthogonal polynomials and Stieltjes transforms.

Among the topics we touch upon are:

- the combinatorics of certain word graphs,
- concentration inequalities,
- the Stieltjes transform,
- Hermite polynomials, Fredholm determinants, Laplace asymptotic method, special functions (Airy, Painleve),
- stochastic calculus, etc.

Desirable Previous Knowledge

There are no essential pre-requisites beyond probability, linear algebra, and complex analysis at undergraduate levels

Introductory Reading

1. G. W. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*. CUP, 2009.

Reading to complement course material

1. G. Blower, *Random Matrices: High Dimensional Phenomena*. CUP, 2009.

Stochastic Networks (M24)

F. Kelly

This course uses stochastic models to shed light on important issues in the design and control of communication networks. Randomness arises in communication systems at many levels: for example, the initiation and termination times of calls in a telephone network, or the statistical structure of the arrival streams of packets at routers in the Internet. How can routing, flow control and connection acceptance algorithms be designed to work well in uncertain and random environments?

The first part of the course will describe a variety of classical models that can be used to help understand the performance of large-scale communication networks. Queueing and loss networks will be studied, as well as random access schemes. Parallels will be drawn with models from physics, and with models of traffic in road networks.

The second part of the course will study congestion control algorithms in the Internet. This is an area of some practical importance, with network operators, hardware and software vendors, and regulators actively seeking ways of delivering new services reliably and effectively. The interplay between end-systems and the network has attracted the attention of economists as well as mathematicians and engineers.

Desirable previous knowledge

Mathematics that will be assumed to be known before the start of the course: Part IB Optimization and Markov Chains. Familiarity with Part II Applied Probability would be useful, but is not assumed.

Introductory reading

A feeling for some of the ideas of the course can be taken from

The mathematics of traffic in networks. In *Princeton Companion to Mathematics* (Edited by Timothy Gowers; June Barrow-Green and Imre Leader, associate editors) Princeton University Press, 2008. 862-870.

Literature

1. B. Hajek *Communication Network Analysis*.
2. P. Robert *Stochastic Networks and Queues*. Springer-Verlag, 2003. Chapter 4.
3. H. Chen and D.D. Yao *Fundamentals of Queueing Networks*. Springer-Verlag, 2001.
4. S. Asmussen *Applied Probability and Queues* - second edition. Springer-Verlag, 2003.
5. R. Srikant *The Mathematics of Internet Congestion Control*. Birkhauser, 2004.
6. S. Shakkottai and R. Srikant *Network Optimization and Control*. Foundations and Trends in Networking, NoW Publishers, 2007.

Advanced Probability (M24)

Perla Sousi

This course aims to cover the advanced topics at the core of research in probability theory. There is an emphasis on techniques needed for the rigorous analysis of stochastic processes such as Brownian motion.

The measure-theoretic formulation of probability will be reviewed at the beginning of the course, but students unfamiliar with this will need to consolidate their understanding with some further reading, say at the level of Part A of Williams' book.

Review of measure and integration

Measurable functions, integration, expectation of random variables; statements of Fatou's lemma, monotone and dominated convergence theorems; product measures and independence, statement of Fubini's theorem.

Conditional expectation

Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

Martingales

Discrete time martingales, submartingales and supermartingales; optional stopping; Doob's inequalities, upcrossings, convergence theorems, backwards martingales. Applications.

Continuous-time random processes

Kolmogorov's criterion, path regularization theorem for martingales; continuous-time martingales.

Weak convergence in \mathbb{R}^n

Definition and characterizations, convergence in distribution, tightness, Prohorov's theorem, characteristic functions, Lévy's continuity theorem.

Sums of independent random variables

Strong law of large numbers, central limit theorem, Cramér's theorem on large deviations.

Brownian motion

Wiener's theorem. Scaling and symmetry properties. Martingales associated to Brownian motion, strong Markov property, reflection principle, hitting times. Sample path properties, recurrence and transience. Brownian motion and the Dirichlet problem. Donsker's invariance principle. Wiener sausage.

Poisson random measures

Definition, compound Poisson processes, Campbell's formula. Infinite divisibility, statement of Lévy-Khinchin theorem. Bartlett's theorem, application to mobile geometric graphs.

Appropriate books

1. R. Durrett, Probability: Theory and Examples, Wadsworth 1991
2. O. Kallenberg, Foundations of Modern Probability, Springer 1997
3. J.F.C. Kingman, Poisson Processes, Oxford Studies in Probability
4. P. Mörters and Y. Peres, Brownian motion, Chapters 1,2,3,5 Cambridge University Press 2010
5. L.C.G. Rogers and D. Williams, Diffusions, Markov processes, and Martingales Vol. I (2nd edition), Chapters I & II. Wiley 1994
6. D.W. Stroock, Probability Theory – An analytic view, Chapters I–V Cambridge University Press 1993
7. D. Williams, Probability with Martingales, Cambridge University Press 1991

Stochastic Calculus and Applications (L24)

J. Norris and A.Sen

The course is an introduction to the theory of continuous-time stochastic process , with an emphasis on the central role played by the Brownian motion. It complements the material in Advanced Probability and Advanced Financial Models.

1. *Review of Brownian motion.* Lévy's construction (Wiener's theorem). Strong Markov property. Sample path properties.
2. *Continuous stochastic calculus.* Adapted and previsible processes. Martingales and local martingales. Martingale inequalities. Quadratic variation and co-variation processes. Itô's isometry and definition of stochastic integral. Itô's formula. Stratonovich integral.
3. *Applications to Brownian motion.* Brownian martingales. Lévy's characterization of Brownian motion. Dubins-Schwartz theorem. Dirichlet problem. Donsker's invariance principle. Cameron-Martin theorem. Girsanov's theorem.
4. *Stochastic Differential Equations.* Strong and weak solutions. Notions of existence and uniqueness. Yamada-Watanabe theorem. Markov property. Relation to second order partial differential equations.
5. *Stroock-Varadhan theory.* Diffusions, martingale problems, equivalence with SDEs, Stroock-Varadhan theorem on diffusion approximation of Markov chains, examples.

Desirable Previous Knowledge

Knowledge of measure theoretic probability at the level of Part III Advanced Probability course will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Introductory Reading

1. R. Durrett (2010) Probability: theory and examples (fourth edition). Cambridge University Press

Reading to complement course material

1. I. Karatzas and S. Shreve. (1998) Brownian motion and stochastic calculus. Springer.
2. D. Revuz and M. Yor. (2001) Continuous time martingales and Brownian motion. Springer.
3. L. C. G. Rogers and D. Williams. (2002) Diffusions, Markov processes, and martingales. Vol. 1 and 2. Cambridge University Press.

Topics on conformal invariance and randomness (L16)

Wendelin Werner

This course is an introduction to the study of continuous random two-dimensional geometric objects.

Study of advanced probability is often divided into the study of stochastic processes indexed by continuous time, such as Brownian motion (and moving further into stochastic differential equations and the relation with PDEs) on the one hand, and discrete lattice-based models from physics (such as percolation or the Ising model) on the other hand.

When the dimension of the space is equal to two, it turns out (this has mostly been pointed out in the last 10-15 years and is nowadays a very active research field) that one can make sense of the continuous

random objects that appear in the scaling limit of the latter discrete models. One key tool for this is complex analysis.

In the present course, we will neither discuss in detail how and why the discrete lattice-based models converge to the continuous objects, nor their relation to conformal field theory or representation theory, but we will define various random continuous structures, discuss their properties and point out some of the close relations between them, that are due to their conformal invariance.

Topics covered are likely to include:

- Conformal invariance of planar Brownian paths, Brownian loops and Brownian excursions.
- The Schramm-Loewner Evolution (SLE): these conformally invariant dynamically defined random curves in the plane have been defined in 1999 by Oded Schramm are the key to understand many properties of all these conformally invariant models.
- Restriction properties, and the relation between outer boundaries of Brownian loops and some SLE. Also, the relation with the scaling limit of percolation clusters (we will a priori not discuss the existence of this scaling limit that had been proved by Smirnov in 2001).
- The Gaussian Free Field and its basic properties, and its relation to SLE.
- The conformal loop ensembles and their relation to all the above.

Pre-requisite Mathematics

The course will be rather self-contained, but complex analysis at undergraduate level (Riemann's mapping theorem etc.) together with some basic background on Brownian motion (up to Itô's formula) is essential. Knowledge of discrete models such as percolation (such has covered in Grimmett's Michaelmas term Part III course) would be very helpful.

Literature

There is a vast and growing literature on SLE. Information on some of the topics that will be discussed can be found in the following lecture notes or books (the latter three can be also downloaded on the preprint server ArXiv). Additional references will be provided during the course.

Gregory F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, 2005.

Wendelin Werner, *Random planar curves and Schramm-Loewner evolutions*, in Lectures on probability theory and statistics, Saint-Flour, Lecture Notes in Math., 1840, Springer, 2004, 107-195 (2004).

Wendelin Werner, *Conformal restriction and related questions*, Probab. Surv. **2** (2005), 145–190 (electronic).

Wendelin Werner, *Some recent aspects of random conformally invariant systems*, Mathematical statistical physics, Elsevier B. V., Amsterdam, 2006, pp. 57–99.

Statistics

Semiparametric Statistics (L16)

I. Castillo

In statistics, semiparametric models are those where the quantity of interest for estimation is a finite-dimensional aspect of the full model, which itself is typically infinite-dimensional. Semiparametrics is currently a very active area of research in statistics, with applications ranging from signal processing to medicine, from econometrics to bioinformatics.

In many relevant situations, notwithstanding the presence of a high-dimensional nuisance parameter, it is possible to find estimators which converge at a parametric $1/\sqrt{n}$ -rate from n observations. A main part of the course will focus on the theory for this case, in an asymptotic framework. The theory will be illustrated with examples from signal processing and medicine, in particular survival analysis and the famous Cox' proportional hazards model.

We will also give some elements of theory for situations where the rate of convergence is slower than $1/\sqrt{n}$, which typically occurs for less regular functionals, a part of the subject which still has many open questions.

Pre-requisite. It is assumed that students have followed introductory courses in probability and statistics.

Applied Statistics. (M16)

Ioana A. Cosma and Robin Evans

This is a *practical* course (2 units: 8 lectures and 8 classes) aiming to develop skills in analysis and interpretation of data, and in writing up of data analyses. Students are strongly encouraged to attend the course Statistical Theory for the theoretical background to the results used in the practical analysis of data.

Introduction to Linux and R on the Statistical Laboratory computing network. Use of LaTeX for report writing.

Exploratory data analysis, graphical summaries. Introduction to non-parametric tests.

The essentials of Generalized Linear Modelling.

Linear regression, orthogonal polynomials, standard experimental designs: factorial experiments and interpretation of interactions.

Regression diagnostics: residuals, leverages and other related plots. Collinearity. Box-Cox transformations.

Discrete data analysis: binomial and Poisson regression. Multi-way contingency tables.

Some special topics, e.g., (i) use of the Akaike Information Criterion for model-search. (ii) quasi-likelihood and over-dispersion.

Desirable Previous Knowledge

It is assumed that you will have done a basic statistics course (including t -tests, χ^2 -tests, F -tests).

Introductory Reading

1. Dobson, A.J. (2002) *An Introduction to Generalized Linear Models*. Chapman & Hall/CRC. 2nd edition.
2. Pawitan, Y. (2001) *In All Likelihood : Statistical Modelling and Inference Using Likelihood*. Oxford Science Publications.

Reading to complement course material

1. Agresti, A. (1990) *Categorical Data Analysis*. Wiley. 2nd edition.
2. McCullagh, P. and Nelder, J.A. (1989) *Generalized Linear Models*. Chapman & Hall. 2nd edition.
3. Venables, W.N. and Ripley, B.D. (2002) *Modern Applied Statistics with S*. Springer-Verlag. 4th edition.

Applied Statistics II (L4+4)

B.D.M. Tom

This is the second part of the Applied Statistics course, which consists of 4 lectures and 4 classes given in the Lent term. This course (i.e. both the first and second parts) will count as a 3 unit (24 lectures) course.

Four topics are to be covered in this part of the course. Previous topic areas consist of non-parametric density estimation; EM algorithm and mixture models; methods for survival data analysis; longitudinal data analysis and multi-state modelling; additive and generalized additive models; parametric approaches to handling over-dispersed count data; and cost-effectiveness analysis.

The four topics chosen will be lectured on and then put into practice using the R statistical software environment. In the practical classes, emphasis is placed on the importance of clear presentation and interpretation of the analysis performed. Students are recommended to submit written solutions to the lecturer for feedback.

Pre-requisite

It is assumed that you have taken the first part of this course in the Michaelmas term and previously done a basic statistics course. Familiarity with using R to read in data, produce graphical and tabular summaries, perform simple hypothesis tests (including t, χ^2 , F and non-parametric tests) and analyses based on linear and generalized linear models is assumed.

Literature

1. Venables, W.N. and Ripley, B.D. (2002). *Modern Applied Statistics with S*. Springer-Verlag (4th Edition).

Reading to complement course material

A list of references will accompany each topic covered.

Time Series and Monte Carlo Inference. (M/L 16)

A. P. Dawid

The course consists of two distinct components, Time Series and Monte Carlo Inference, each with 8 lectures. You must take the two components together for the examination.

Time Series

Time series analysis refers to problems in which observations are collected at regular time intervals and there are correlations among successive observations. Applications cover virtually all areas of Statistics but some of the most important include economic and financial time series, and many areas of environmental or ecological data. This course will cover some of the most important methods for dealing with these problems. In the case of time series, these include the basic definitions of autocorrelations, etc., time-domain fitting including auto-regressive and moving average processes, and spectral methods.

Literature: Time series

1. Chatfield, C. *The Analysis of Time Series: An Introduction*, Chapman and Hall (1975).
2. Brockwell, P. J. and Davis, R. A. *Time Series: Theory and Methods*. Springer Series in Statistics (1986).
3. Brockwell, P. J. and Davis, R. A. *Introduction to Time Series and Forecasting*. Springer (1996).
4. Diggle, P. J. *Time Series: A Biostatistical Introduction*. Oxford University Press (1990).
5. Shumway, R. H. and Stoffer, D. S. *Time Series Analysis and its Applications (with R examples)*. Springer Texts in Statistics (2006).

Monte Carlo Inference

Monte Carlo methods are concerned with the use of stochastic simulation techniques for statistical inference. These have had an enormous impact on statistical practice, especially Bayesian computation, over the last 20 years, due to the advent of modern computing architectures and programming languages. This course covers the theory underlying some of these methods and illustrates how they can be implemented and applied in practice.

The following topics will be covered. Techniques of random variable generation. Markov chain Monte Carlo (MCMC) methods for Bayesian inference. Gibbs sampling, Metropolis–Hastings algorithm, reversible jump MCMC.

Literature: Markov Chain inference

1. Gentle, J. E. *Random Number Generation and Monte Carlo Methods*, Second Edition, Springer (2003)
2. Ripley, B. D. *Stochastic Simulation*, Wiley (1987)
3. Gamerman, D. and Lopes, H. F. *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference*, Second Edition, Chapman and Hall (2006)
4. Robert, C. P. and Casella, G. *Monte Carlo Statistical Methods*, Springer (1999)

Desirable Previous Knowledge

You should have attended introductory Probability and Statistics courses. A basic knowledge of Markov chains would be helpful. Prior familiarity with a statistical programming package such as R or Matlab would also be desirable.

Statistical Theory (L24)

Richard Nickl

This lecture course covers, with complete proofs, selected core topics of modern mathematical statistics. We shall treat both finite dimensional ('parametric') models and infinite-dimensional ('nonparametric') models. Next to classical concepts from statistical inference the mathematics involved includes elements from probability theory, analysis and approximation theory.

The course starts with a quick review of basic concepts of probability theory, and develops some basic tools from empirical process theory that will be useful throughout.

Subsequently the classical theorems about the asymptotic distribution of maximum likelihood estimators and of the likelihood ratio test statistic in general parametric models will be derived, under weak assumptions. We shall also develop the main ideas behind Le Cam's notion of local asymptotic normality of experiments, and use it to investigate the large-sample behavior of Bayesian estimators in parametric models (Bernstein-von Mises theorem).

The third section of this course gives an introduction to the theory of statistical inference in infinite-dimensional models – 'nonparametric statistics'. Fundamental problems such as the estimation of a distribution function, a density function or a regression function, will be considered. The research area of nonparametric statistics has been evolving rapidly in the last 15 years, and it has produced a variety of deep and surprising mathematical results, as well as some fascinating open problems.

Section I: Review of Probability Theory, basic empirical process theory and uniform laws of large numbers. [2 lectures]

Section II: Finite-Dimensional Models: Maximum likelihood estimators, likelihood ratio and Wald test statistic, asymptotic distribution theory, efficiency, Delta-method, locally asymptotically normal models, Bayesian estimators and the Bernstein-von Mises theorem, examples and applications to linear and generalized linear models. [7]

Section III: Infinite-Dimensional Models: estimation of a probability distribution function (Donsker and Kolmogorov-Smirnov theorems), minimax rates of convergence, basic approximation theory (convolution with kernels, linear approximation in Hilbert spaces, wavelet theory), density estimation (kernel- and wavelet estimators; confidence sets), nonparametric regression (kernel-, local polynomial-, natural cubic spline- and wavelet regression), inverse problems. [15]

Pre-requisite Mathematics

Having had first basic courses in statistics, probability and analysis is required, but not much more: Some basic knowledge of measure theory and linear/functional analysis is useful, but this can certainly be acquired during the course without any difficulties. What is generally needed is the interest and ability to synthesize several branches of mathematics.

Literature

There are complete lecture notes on my website. Further relevant reading is:

1. Tsybakov, A.B. Introduction to nonparametric estimation, Springer 2009.
2. Van der Vaart, A.W. Asymptotic statistics, Cambridge University Press 1998.

Actuarial Statistics (L16)

S.M. Pitts

This course provides an introduction to various topics in non-life insurance mathematics. These topics feature mainly in the Institute and Faculty of Actuaries examination CT6.

Topics covered in lectures include

1. Loss distributions
2. Reinsurance
3. Aggregate claims
4. Ruin theory
5. Credibility theory
6. No claims discount systems

Prerequisite mathematics

This course assumes

an introductory probability course (including moment generating functions, probability generating functions, conditional expectations and variances)

a statistics course (including maximum likelihood estimation, Bayesian methods)

that you know what a Poisson process is

that you have met discrete time finite statespace Markov chains

Backup to the lectures

Lectures will be supplemented with examples sheets and examples classes.

Literature

1. S. Asmussen *Ruin Probabilities*. World Scientific, 2000.
2. C.D. Daykin, T. Pentikäinen and E. Pesonen, *Practical Risk Theory for Actuaries and Insurers*. Chapman and Hall, 1993.
3. D.M. Dickson, *Insurance Risk and Ruin*. CUP, 2005.
4. J. Grandell, *Aspects of Risk Theory*. Springer, 1991.
5. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance*. Wiley, 1999.

Biostatistics (L16)

This course consists of two components: Survival Data and Statistics in Medical Practice. Together these make up one 2 unit (16 lecture) course. You must take both components together for the examination. survival Data has 10 lectures and 2 classes; Statistics in Medical Practice has 6 lectures and 1 class.

Survival Data (L10+2)

P. Treasure

6 Lectures on Fundamentals of Survival Analysis

Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

4 Lectures on Current Topics in Survival Analysis

In recent years there have been lectures on: frailty, cure, relative survival, empirical likelihood, counting processes and multiple events.

2 Example Classes

Following immediately after the lectures, the example classes apply the lectured material to real survival analysis contexts and datasets.

1 Revision Class (2 hours)

The revision class takes place just before the examination period in the Easter Term.

Level:

General

Principal book

1. D. R. Cox & D. Oakes, *Analysis of Survival Data*, London: Chapman & Hall (1984).

Other books

2. P. Armitage, J. N. S. Matthews & G. Berry, *Statistical Methods in Medical Research* (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].
3. M. K. B. Parmar & D. Machin, *Survival Analysis: A Practical Approach* (1995), Chichester: John Wiley.

Statistics in Medical Practice (L6)

Sheila M . Bird, Rebecca Turner and Ian White.

Each lecture will be a self-contained study of a topic in biostatistics, which may include study design (including randomization and evaluation of interventions), meta-analysis of clinical trials and/or non-randomized studies, and statistical criticism of published medical papers. The relationship between the medical issue and the appropriate statistical theory will be illustrated.

Level:

General

Appropriate books

There are no appropriate books, but relevant medical papers will be made available beforehand for prior reading. It would be very useful to have some familiarity with media coverage of medical stories involving statistical issues, e.g. from Behind the Headlines on the NHS Choices website:

<http://www.nhs.uk/News/Pages/NewsIndex.aspx>

Operational Research and Mathematical Finance

Mathematics of Operational Research (L24)

F. Fischer

This course is accessible to a candidate with mathematical maturity who has no previous experience of operational research; however it is expected that most candidates will already have had exposure to some of the topics listed below.

- Lagrangian sufficiency theorem. Strong Lagrangian problems. Supporting hyperplane theorem. Sufficient conditions for convexity of the optimal value function. Fundamentals of linear programming. Dual linear program. Shadow prices. [2]
- Simplex algorithm. Two-phase method. Dual simplex algorithm. Gomory's cutting plane method. [3]
- Complexity of algorithms. NP-completeness. Exponential complexity of the simplex algorithm. Polynomial time algorithms for linear programming. [2]
- Network simplex algorithm. Transportation and assignment problems, Ford-Fulkerson algorithm, max-flow/min-cut theorem. Shortest paths, Bellman-Ford algorithm, Dijkstra's algorithm. Minimum spanning trees, Prim's algorithm. MAX CUT, semidefinite programming, interior point methods. [5]
- Branch and bound. Dakin's method. Exact, approximate, and heuristic methods for the travelling salesman problem. [3]
- Cooperative and non-cooperative games. Two-player zero-sum games. Existence and computation of Nash equilibria, linear complementarity problems, Lemke's algorithm. Bargaining. Coalitional games, core, nucleolus, Shapley value. Mechanism design, Arrow's theorem, Gibbard-Satterthwaite theorem, VCG mechanism. Auctions, revenue equivalence, optimal auctions. [9]

Books

1. M.S. Bazaraa, J.J. Jarvis and H.D. Sherali: Linear Programming and Network Flows, Wiley (1988).
2. D. Bertsimas, J.N. Tsitsiklis. Introduction to Linear Optimization. Athena Scientific (1997).
3. N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani. Algorithmic Game Theory. Cambridge University Press (2007).
4. M. Osborne, A. Rubinstein: A Course in Game Theory. MIT Press (1994).

Advanced Financial Models (M24)

L.C.G. Rogers

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability, Stochastic Calculus and Applications, and Optimal Investment.

- One-period models. Arbitrage and equivalent martingale measures. Attainable claims and market completeness.
- Multi-period discrete time models. Filtrations and martingales. European and American claims. Optimal stopping.

- Brownian motion and stochastic calculus. Brownian motion and its quadratic variation. Stochastic integration. Girsanov's theorem. Itô's formula. Martingale representation theorem.
- Black–Scholes model and generalizations. Admissible strategies. Equivalent martingale measures. Pricing and hedging in Markovian models. The Black–Scholes model. Local and stochastic volatility models.
- Interest rate models. Short rates, forward rates, and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.

Pre-requisite Mathematics

A knowledge of probability theory at the level of Part II Probability and Measure will be assumed. Familiarity with Part II Stochastic Financial Models is helpful.

Literature

Lecture notes will be distributed. Additionally, the following books may be helpful.

1. M. Baxter & A. Rennie. (1996) Financial calculus: an introduction to derivative pricing. Cambridge University Press
2. M. Musiela and M. Rutkowski. (2006) Martingale Methods in Financial Modelling. Springer.
3. I. Karatzas. (1997) Lectures on the mathematics of finance. American Mathematical Society
4. Lamberton, D. & B. Lapeyre. (1996) Introduction to stochastic calculus applied to finance. Chapman & Hall
5. S. Shreve. (2005) Stochastic Calculus for Finance: Vol. 1 and 2. Springer-Finance

Optimal Investment. (L16)

M. Tehranchi

The course will study a wide range of optimal investment/consumption problems. Examples will be drawn from both discrete- and continuous-time financial models. The following provisional list of topics indicates some of the intended content.

- self-financing portfolios and the wealth equation,
- the dynamic programming principle and the Bellman equation,
- martingale methods and Lagrangian duality,
- mutual fund theorems,
- the Hamilton–Jacobi–Bellman partial differential equation,
- utility indifference pricing,
- market equilibrium models,
- backward stochastic differential equations,
- forward utility functions,
- optimisation under parameter uncertainty

Desirable Previous Knowledge

A firm grasp of martingale theory and a working knowledge of stochastic calculus will be assumed. The course complements Part III Advanced Financial Models.

Introductory Reading

Lecture notes will be distributed. Additionally, the following books may be helpful.

1. D. Kennedy. (2010) Stochastic Financial Models. Chapman & Hall/CRC Financial Mathematics Series.
2. I. Karatzas and S. and Shreve. (1998) Methods of Mathematical Finance. Springer.

Particle Physics, Quantum Fields and Strings

The courses on Symmetry and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of Symmetry and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p . Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, ‘Golden Rule’ for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ -functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

<http://www.maths.cam.ac.uk/undergrad/schedules/>

Year	Courses
Second	<i>Essential:</i> Quantum Mechanics, Special Relativity, Methods, Complex Methods. <i>Helpful:</i> Electromagnetism.
Third	<i>Essential:</i> Principles of Quantum Mechanics, Classical Dynamics. <i>Very helpful:</i> Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

Supersymmetry (L16)

B.C. Allanach

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Three examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model and have a rough knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course.

Reading to complement course material

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as I know) Wess and Bagger.

1. A supersymmetry primer, S.P. Martin, [hep-ph/9709256](#) is good for phenomenological aspects
2. Introduction to supersymmetry, J.D. Lykken, [hep-th/9612114](#). This introduction is good for extended supersymmetry and more formal aspects
3. Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics
4. Introduction to Supersymmetry and Supergravity, P West, World Scientific Publishing Co. (1990) is one of the standard references
5. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this book has the opposite sign of metric to the course.
6. Videos of a similar lecture course: <http://users.hepforge.org/allanach/teaching.html>

Quantum Field Theory (M24)

Professor A. C. Davis

Quantum Field Theory is the language in which all of modern physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is developed in terms of operators which create and annihilate particles and anti-particles and the associated Fock space of quantum physical states is explained.

Interactions are introduced using perturbative techniques and the role of Feynman diagrams is explained. This is first illustrated for theories with a purely scalar field interaction, and then for a Yukawa coupling between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

Necessary Previous Knowledge

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Books

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
2. L.H. Ryder, *Quantum Field Theory*, Cambridge University Press (1996).
3. S. Weinberg, *The Quantum Theory of Fields Vol I*, Cambridge University Press (1995)
4. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2003)

Solitons and Instantons. (E16)

Nick Dorey

In classical non-linear field theory, solutions which are stable, localized in space and particle-like are called solitons. Solutions that are also localised in Euclidean time are called instantons. In certain cases, second order field equations can be reduced to first order Bogomolny equations, which makes it easier to analyse these solutions. The resulting equations are often integrable in some sense, and explicit solutions can sometimes be found. The existence of such classical solutions often has profound implications for the corresponding quantum theory such as the phenomenon of strong-coupling/weak-coupling duality. In this course we will develop these ideas in the context of integrable models in two spacetime dimensions and also discuss their applications to non-abelian gauge theories in four dimensions.

Desirable Previous Knowledge

Quantum Field Theory. Advanced Quantum Field Theory

Introductory Reading

1. T.D. Lee (1981) Particle physics and introduction to field theory, Harwood. Chapter 7.

Reading to complement course material

1. R. Rajaraman (1982) Solitons and instantons, An introduction to solitons and instantons in quantum field theory. North-Holland Publishing Co.
2. N.S. Manton and P.M. Sutcliffe (2004) Topological Solitons, CUP.

Symmetries and Particles (M24)

Michael B. Green

The course provides introductory material on properties of groups and their applications to particle physics. In particular Lie groups and Lie algebras, and their representations in terms of matrices, are discussed in some detail. The examples of $SU(2)$ and $SU(3)$ play a particularly important rôle. The observed elementary particles and their interactions are first reviewed. The particles are divided into hadrons and leptons. Hadrons are characterised by having strong interactions and are composite particles made up of quarks. Hadrons and leptons also undergo electromagnetic and weak interactions, which are responsible for their decay. Various quantum numbers which distinguish particles and their interactions are described.

Basic properties of Lie groups and Lie algebras are introduced. The geometric structures underlying the theory of Lie groups and algebras are presented, and properties of representations explained. Various aspects of angular momentum, which is associated with the group $SU(2)$, are derived. Similarly, properties of $SU(3)$ are examined in some detail. This is applied to hadrons, which are classified in terms of representations of global symmetry groups such as $SU(2)$ and $SU(3)$. This motivates the description of their

states in terms of quarks. The role of Lie groups in understanding spacetime symmetry is also discussed and the Lorentz group and Poincaré group are described. This leads to consideration of Dirac matrices and spinors. Lie groups also play an essential role in non-abelian gauge theories. Local symmetry requires vector gauge fields, and the corresponding quantum field theories are the basis of modern particle physics. The weak and electromagnetic interactions are described by the “electro-weak” theory with gauge group $SU(2) \times U(1)$, while the strong interaction is described by “QCD”, which has the gauge group $SU(3)$. At the end of the course, the systematic classification of simple Lie algebras is reviewed. The notion of roots and weights and their properties are explained and their important role in the classification of representations is presented. The emphasis here is on the essential ideas rather than mathematical proofs.

Desirable Previous Knowledge

Familiarity with the treatment of angular momentum in quantum mechanics, the states $|jm\rangle$ and the action of the operators J_3 , J_{\pm} on them. Understanding how products of angular momentum states can be combined using Clebsch-Gordan coefficients is also important.

Introductory Reading

1. Perkins, D.H., Introduction to High energy Physics, 4th ed. CUP (2000).

Reading to complement course material

1. Georgi, H., Lie Algebras in Particle Physics, Westview Press (1999).
2. Fuchs, J, and Schweigert C., Symmetries, Lie Algebras and Representations, 2nd ed. CUP (2003).
3. Jones H.F., Groups, Representations and Physics, 2nd ed., IOP Publishing (1998).

Statistical Field Theory (M16)

R R Horgan

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory, and the construction of effective field theories. The discussion is concerned with statistical systems including their relationship with quantum field theory in its Euclidean formulation.

The phenomenology of phase transitions is reviewed, leading to the introduction of the theory of critical phenomena. Landau-Ginsburg theory and mean field theory are presented and applied to the Ising model. The classification of phase transitions and their relationship with critical points is presented, and the renormalization group is introduced first in the context of the soluble 1D Ising model and then in general. The renormalization group is used for calculating properties of systems near a phase transition, for example in the Ising and Gaussian models, and the concepts of critical exponents, anomalous dimensions, and scaling are discussed.

The idea of the continuum limit for models controlled by a critical point and the relationship with continuum quantum field theory is elucidated.

Perturbation theory is introduced for the scalar field model with interactions and some example calculations are presented.

A background knowledge of Statistical Mechanics at an undergraduate level is essential. Although not a formal prerequisite, attendance at the Part III course Quantum Field Theory is a considerable advantage and strongly advised.

Books

1. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Oxford Scientific Publishing (1992)
2. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996)
3. J Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics (1996)
4. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992)
5. D Amit and V Martín-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005)
6. C Itzykson and J-M Drouffe, *Statistical Field Theory*, Vols. 1-2, Cambridge University Press (1991)

Advanced Quantum Field Theory (L24)

H. Osborn

Quantum field theory (QFT) is the basic theoretical framework for describing elementary particles and their interactions (excluding gravity) and is essential in the understanding of string theory. It is also used in many other areas of physics including condensed matter physics, astrophysics, nuclear physics and cosmology. The Standard Model, which describes the basic interactions of particle physics, is a particular type of QFT known as a gauge theory. Gauge theories are invariant under symmetry transformations defined at each point in spacetime which form a Lie Group under composition. To quantise a gauge theory, it is necessary to eliminate non-physical degrees of freedom and this requires additional theoretical tools beyond those developed in the introductory quantum field theory course.

A variety of new concepts and methods are first introduced in the simpler context of scalar field theory. The functional integral approach provides a formal non-perturbative definition of any QFT which also reproduces the usual Feynman rules. The course discusses in a systematic fashion the treatment of the divergences which arise in perturbative calculations. The need for regularisation in QFT is explained, and the utility of dimensional regularisation in particular is emphasised. It is shown how renormalisation introduces an arbitrary mass scale and renormalisation group equations which reflect this arbitrariness are derived. Various physical implications are then discussed.

The rest of the course is concerned specifically with gauge theories. The peculiar difficulties of quantising gauge fields are considered, before showing how these can be overcome using the functional integral approach in conjunction with ghost fields and BRST symmetry. A renormalisation group analysis reveals that the coupling constant of a quantum gauge theory can become effectively small at high energies. This is the phenomenon of asymptotic freedom, which is crucial for the understanding of QCD: the gauge theory of the strong interactions. It is then possible to perform perturbative calculations which may be compared with experiment. Further properties of gauge theories are discussed, including the possibility that a classical symmetry may be broken by quantum effects, and how these can be analysed in perturbation theory. Such anomalies have important implications for the way in which gauge particles and fermions interact in the Standard Model.

Desirable Previous Knowledge

Knowledge of the Michaelmas term course "Quantum Field Theory" will be assumed. Familiarity with the content of "Symmetry and Particle Physics" (M24) would be very helpful.

Introductory Reading

1. Section 9.1 and Chapter 15 of *An Introduction to Quantum Field Theory*, Peskin M E and Schroeder D V (Addison-Wesley 1996)

Reading to complement course material

1. Quantum Field Theory, Ryder L H (2nd edn CUP 1996)
2. An Introduction to Quantum Field Theory, Peskin M E and Schroeder D V (Addison-Wesley 1996)
3. Quantum Theory of Fields, Vols. 1 & 2, Weinberg S (CUP 1996)

String Theory (L24)

Malcolm Perry

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework.

While string theory is often paraded as the ultimate theory of everything, a less trumpeted facet is the way in which the theory reveals insights and connections between other, seemingly unrelated, aspects of physics. Much of modern research in theoretical physics uses string theory as a tool to understand more down-to-earth physical systems, most notably strongly coupled quantum field theories.

This course will provide an introduction to various topics in string theory. We aim to cover how to construct both the bosonic string and the superstring and examine the spectrum of states in these theories. We then aim to look at some simple calculations of scattering amplitudes. If time permits, we will then move on and examine some more interesting topics: how it is that string theory encompasses general relativity, D-brane technology and how one starts to construct models of elementary particles starting from string theory.

Desirable Previous Knowledge

You will need knowledge of the material from the Quantum Field Theory course and, as it progresses, the Advanced Quantum Field Theory course. You should be comfortable with general relativity.

Reading to complement course material

1. M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory," Cambridge University Press (1987).
2. E. Kiritsis, "String Theory in a Nutshell," Princeton University Press, (2007).
3. K. Becker, M. Becker and J.H. Schwarz, "String Theory and M-Theory: A Modern Introduction," Cambridge University Press, (2007).

Standard Model (L24)

M.B. Wingate

The Standard Model of particle physics is, by far, the most successful application of quantum field theory. As this booklet goes to press, this model accurately describes all elementary particle physics measurements involving strong, weak, and electromagnetic interactions.

The Standard Model is the quantum theory of the gauge group $SU(3) \times SU(2) \times U(1)$ with fermion fields for the leptons and quarks. The course aims to demonstrate how this model is realised in nature and to point out some of its limitations as a complete theory of Nature. It is intended to complement the more general Advanced QFT course.

This course begins by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content in terms of spin 1/2 leptons and quarks and also the spin 1 gauge bosons. The parity P , charge conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in Nature; e.g. only left-handed particles feel the

weak force in violation of parity symmetry. It is shown how CP violation becomes possible when there are three generations of particles.

Ideas of spontaneous symmetry breaking are developed. The spectrum of strongly bound states of light quarks is due to the global flavour symmetry of the strong interactions being spontaneously broken. As described by the Higgs Mechanism, the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry.

We show how to obtain cross-sections and decay rates from the matrix element squared of a process. Various scattering and decay processes can be calculated in the electroweak sector using perturbation theory because of the smallness of the couplings. We discuss the topic of neutrino masses and oscillations, an important window into physics beyond the Standard Model.

The strong interactions are based upon the gauge theory with (unbroken) gauge group $SU(3)$, called quantum chromodynamics (QCD). At low energies quarks are confined, forming bound states called hadrons. In such a non-abelian theory, the coupling constant decreases in higher energy processes to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. We also examine inelastic scattering of leptons on hadrons. At leading order the latter process can be described in terms of free quarks in the hadron, historically providing strong evidence for their existence.

Throughout the course, we touch upon open questions. Very high energy experiments and very precise experiments are currently striving to observe effects not describable by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them. It is advantageous to attend the Advanced Quantum Field Theory course during the same term as attending this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
2. T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press (1984).
3. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press (1994).
4. I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics*, IoP Publishing (1989).
5. F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, John Wiley and Sons (1984).

Relativity and Gravitation

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ -function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

<http://www.maths.cam.ac.uk/undergrad/schedules/>

Year	Courses
Second	<i>Essential:</i> Methods, Special Relativity, Principles of Dynamics, Quantum Mechanics. <i>Helpful:</i> Electromagnetism, Geometry, Complex Methods.
Third	<i>Very helpful:</i> General Relativity, Statistical Physics, Electrodynamics, Methods of Mathematical Physics.

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

Discrete Quantum Gravity. (L16)

Non-Examinable (Graduate Level)

Benjamin Bahr

In the first part of this (non-examinable) course, we will discuss the motivation for discretising gravity and then introduce a discretisation scheme for General Relativity, Regge calculus, illustrating its use in both classical and quantum gravity. Regge calculus forms the basis for much other work in discrete gravity and quantum geometry. In the rest of the course we will develop a quantization scheme for the geometric degrees of freedom of Regge calculus, called Spin Foam Models. The main aim of these theories is to provide a Feynman path integral for General Relativity. Building upon simplicial decompositions of space-time, we will derive state-sum models for General Relativity from the BF- and Palatini formulations of GR. This will make heavy use of the discrete structures defined in the first part of the course. In 3 dimensions, this will give the Ponzano-Regge model, which is a path integral for 3d Euclidean gravity. In 4d, we will obtain the Barrett-Crane model. We will discuss these models, comparing them with the recently-developed Spin Foam models in 4d, such as the Engle-Pereira-Rovelli model, and seeing how the models give rise, in the semi-classical limit, to the Regge action. Time permitting, we will illustrate the connections to loop quantum gravity and matrix models.

Desirable Previous Knowledge

Basics of General Relativity (part II level) and Quantum Field Theory (in particular the concept of a path integral), as well as Differential Geometry are essential for the course. Concepts from Representation Theory and Algebraic Topology are helpful, but not strictly necessary.

Introductory Reading

1. Rovelli, C. *Quantum Gravity*. CUP.

Reading to complement course material

1. Williams, R M, and Tuckey, P A. *Regge Calculus: a Brief Review and Bibliography*. Classical and Quantum Gravity 9, 1409 (1992).
2. Baez, J. *An Introduction to Spin Foam Models of Quantum Gravity and BF Theory*. Lecture Notes in Physics 543, 25-94 (2000).
3. Barrett, J W, and Crane, L. *A Lorentzian Signature Model for Quantum General Relativity*. Classical and Quantum Gravity 17, 3101-3118 (2000). 66
4. Oriti, D. (Ed.).
5. *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*. CUP.
6. Thiemann, T. *Modern Canonical Quantum General Relativity*. CUP

Cosmology (M24)

J.D. Barrow, A. Challinor

Over the past fifteen years, cosmology has made dramatic advances. A flood of data has transformed our understanding of the basic parameters of the universe – the expansion rate, the densities of various types of energy and the nature of the primordial density variations. The basic big bang picture, underpinned by general relativity, continues to hold good, explaining the expansion of the universe, the cosmic microwave background radiation, the synthesis of light chemical elements and the formation of stars, galaxies and large-scale structures. However, there are important gaps in our understanding including the nature of the dark matter, the cause of the observed late-time acceleration of the universe and the classic puzzle of the initial singularity.

This course will develop the standard big bang cosmology and review its major successes and some of the challenges now faced at the cutting-edge of the field. In particular, the following topics will be covered.

The homogeneous and isotropic universe

Newtonian cosmologies. Review of general relativity. The geometry and dynamics of Friedmann-Robertson-Walker models; photon propagation; distances, ages and properties of cosmological models.

Thermal history and inflation

Dynamics of massive particles in an expanding universe; thermal relics; cosmic neutrinos and the microwave background; big bang nucleosynthesis. The problems of the standard cosmological model and the inflationary universe.

Cosmological perturbations

Statistics of random fields; perturbations in Newtonian theory and an introduction to relativistic perturbation theory; the linear-theory matter power spectrum; simple treatment of cosmic microwave background anisotropies.

Fluctuations from inflation

Quantisation of a massless scalar field in deSitter space; slow-roll inflation and the primordial power spectrum.

Desirable Previous Knowledge

Familiarity with special relativity and statistical physics is essential. Some familiarity with nuclear and particle physics would also be helpful. Students who have not studied general relativity will find it beneficial to attend the Part-III General Relativity course, though this is not essential as some introductory material on general relativity will be reviewed at the start of the course.

Introductory Reading

1. Liddle, A. An Introduction to Modern Cosmology. Wiley (2003)

Reading to complement course material

1. Dodelson, S. Modern Cosmology. Academic Press (2003) *
2. Liddle, A.R. and Lyth, D.H. Cosmological Inflation and Large-Scale Structure. Cambridge (2000) *
3. Liddle, A.R. and Lyth, D.H. The Primordial Density perturbation. Cambridge (2009)
4. Kolb, E.W. and Turner, M.S. The Early Universe. Addison-Wesley (1990) *
5. Weinberg, S. Gravitation and Cosmology. Wiley (1972)
6. Weinberg, S. Cosmology. Oxford (2008) *
7. Peacock, J.A. Cosmological Physics. Cambridge (2000)
8. Mukhanov, V. Physical Foundations of Cosmology. Cambridge (2005)

Books denoted with a * are particularly recommended for this course.

Inflation (M12)

Non-Examinable (Graduate Level)

Daniel Baumann

This course is about methods of theoretical physics and their applications to particle physics and early universe cosmology. The goal is to bring students to the forefront of current research and provide them with tools that will be valuable for their own research. Much of the material of the course is selected on the “I wish I had learned that earlier” basis. Although we will use inflation as our main example, much of the techniques that we will develop (such as effective field theory, supersymmetry, etc.) have a much wider range of applicability. I hope that this will make the course of interest to students with a variety of backgrounds from particle physics to cosmology.

The course divides into two parts:

- in the first half we will develop the essentials of cosmological inflation. We will focus most of our efforts on the intricacies of quantum field theory in curved spacetimes. This will culminate in a detailed derivation of the primordial density fluctuations predicted by inflation. Some time will also be spent on the in-in formalism and non-Gaussianities.
- the second half of course will cover a selection of more advanced topics of broad importance in theoretical physics, such as *effective field theory*, *supersymmetry*, the physics of *Goldstone bosons* and *symmetry breaking*. We will illustrate these concepts with applications to particle physics and cosmology and explain their significance in research on inflation.

Desirable Previous Knowledge

Basic knowledge of general relativity, quantum field theory and cosmology will be helpful. However, the course will be presented in a self-contained way, so students with less experience in any of these fields should have no problem to catch up.

Introductory Reading

1. Baumann, *TASI Lectures on Inflation*, arXiv:0907.5424.

Reading to complement course material

1. Mukhanov, *Physical Foundations of Cosmology*.
2. Dodelson, *Modern Cosmology*.
3. Weinberg, *Cosmology*.
4. Mukhanov and Winitzki, *Introduction to Quantum Effects in Gravity*.
5. Peskin, *Introduction to Quantum Field Theory*.
6. Burgess and Moore, *A Standard Model Primer*.
7. Terning, *Modern Supersymmetry*.
8. Skiba, *TASI Lectures on Effective Field Theory*, arXiv: 1006.2142.
9. Goldberger, *Les Houches Lectures on Effective Field Theories*, arXiv:hep-ph/0701129.
10. Burgess, *Goldstone Bosons in Nuclear, Particle and Condensed Matter Physics*, arXiv:hep-th/9808176.
11. Maldacena, *Non-Gaussian Features of Primordial Fluctuations in Single-Field Inflationary Models*, arXiv:astro-ph/0210603.

Applications of General Relativity (L16)

Irena Borzým

This course will introduce some useful mathematical tools and discuss some of the most important applications in general relativity.

An outline of the course is as follows.

1. Two component spinors, their algebra and interpretation.
2. Spinors and Vectors: The relationship with vectors and translating between the two descriptions.
3. Petrov Classification:
4. Shear and focusing:
5. Goldberg-Sachs:
6. Plane and Plane fronted waves.
7. Asymptopia for Minkowski space.
8. Asymptotic simplicity.
9. Conformal transformation formulae.
10. Scri and peeling.
11. A choice of conformal gauge.
12. A spin basis for scri.
13. Asymptotic symmetry.
14. Spin weighted spherical harmonics.
15. Asymptotic solution of the field equations.
16. Bondi 4-momentum and positivity of Bondi mass.
17. Quasilinear first order hyperbolic equations.
- 18 The Cauchy problem.
- 19 Characteristic initial value problem.
- 20 Conformal vacuum field equations of GR
- 21 Wavefronts and wavefront families (if time.)

Note: For completeness of the notes some of the material will be non examinable. These sections will be clearly indicated during the lectures.

Classes There will be three problem sets and three classes of one hour each.

Desirable Previous Knowledge

The general relativity course from the Michaelmas term is a prerequisite.

Introductory Reading

No introductory reading is required, but a casual read through the following may be helpful.

1. J.M. Stewart, *Advanced General Relativity*. CUP, 1993. Introductory Section on Spinors.
2. R. Penrose and W. Rindler, *Spinors and Spacetime Volume 1*, CUP 1988. Introductory Section on Spinors.

Reading to complement course material

J. M. S. Is the best reference text for this course.

1. J.M. Stewart, Advanced General Relativity. CUP, 1993.
2. R. Penrose and W. Rindler, Spinors and Spacetime Volume 1, CUP 1988
3. C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation. Freeman, 1973.
4. R.M. Wald, General Relativity. Chicago UP, 1984.
5. S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime. CUP, 1973.

Applications of Differential Geometry to Physics. (L16)

Maciej Dunajski

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Differential Forms and Vector Fields.
 1. One parameter groups of transformations.
 2. Vector fields and Lie brackets.
 3. Exterior algebra.
 4. Hodge Duality.
- Geometry of Lie Groups.
 1. Group actions on manifolds.
 2. Homogeneous spaces and Kaluza Klein theories.
 3. Metrics on Lie Groups.
- Fibre bundles and instantons.
 1. Principal bundles and vector bundles.
 2. Connection and Curvature.
 3. Twistor space.

Desirable Previous Knowledge

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Reading to complement course material

1. http://www.damtp.cam.ac.uk/research/gr/members/gibbons/gwgPartIII_DGeometry2011-1.pdf
2. Flanders, H. Differential Forms. Dover
3. Dubrovin, B., Novikov, S. and Fomenko, A. Modern Geometry. Springer
4. Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393
5. Arnold. V. Mathematical Methods of Classical Mechanics. Springer.
6. Dunajski. M. Solitons, Instantons and Twistors. OUP.

General Relativity (M24)

H.S. Reall

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

Course website: www.damtp.cam.ac.uk/user/hsr1000

Desirable Previous Knowledge

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then it is strongly recommended that you study an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \rightarrow R^n$, and the Euler-Lagrange equations will be assumed.

Introductory Reading

1. Gravity: An introduction to Einstein's General Relativity, J.B. Hartle, Addison-Wesley, 2003.
2. Relativity: Special, General, and Cosmological, 2nd ed., W. Rindler, OUP, 2006.

Reading to complement course material

There are many excellent books on General Relativity. The following is an incomplete list:

1. General Relativity, R.M. Wald, Chicago UP, 1984.
2. Spacetime and geometry: an introduction to General Relativity, S.M. Carroll, Addison-Wesley, 2004.
3. Advanced General Relativity, J.M. Stewart, CUP, 1993.
4. Gravitation and Cosmology, S. Weinberg, Wiley, 1972.

Our approach will be closest to that of Wald. Carroll's book is a very readable introduction. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Advanced Cosmology (L24)

E.P.S. Shellard and E.A. Lim

This course will take forward at much greater depth the topics in modern cosmology covered at the end of the Michaelmas Term course. The prediction from fundamental theory for the primordial perturbation spectrum remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave sky. This course will develop the mathematical tools and physical understanding necessary for research in this very active area. If time permits we will also consider applications for specific models of particular current interest*.

Cosmological Perturbation Theory

- The 3+1 formalism and the Einstein equations
- Linearised Einstein equations for an expanding universe

- Review of density perturbation theory, transfer functions etc

Cosmic Microwave Sky

- Relativistic kinetic theory
- Collisionless Boltzmann equation
- Photon scattering and diffusion
- The CMB temperature power spectrum
- CMB Polarization

Topical issues: Non-Gaussianity from Inflation, Gravitational Waves

- “In-in” formalism and higher order correlation functions
- Non-Gaussianities from inflation
- Prospects for observations of non-Gaussianity
- Cosmological sources of stochastic gravitational waves
- Detection of cosmological gravitational waves
- Signatures of brane inflation and extra dimensions*

Pre-requisites

Familiarity with introductory Quantum Field Theory is recommended.

Course texts

Mukhanov, V. *Physical Foundation of Cosmology*, Cambridge (2005).

Bardeen, J.M., *Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure*, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request).

Chen, X., *Primordial Non-Gaussianities from Inflation Models*, arxiv:1002.1416.

Dodelson, S., *Modern Cosmology*, Elsevier 2003.

Weinberg, S. *Cosmology*, Oxford University Press 2008.

Useful references

Efstathiou, G., in *Physics of the Early Universe*, Proc. 36th Scottish Summer School, eds. J. Peacock, et al., p. 361, Adam Hilger (1990).

Kolb, E.W. & M.S. Turner, *The Early Universe*, Addison-Wesley (1990).

Liddle, A. & Lyth, D., *Cosmological Inflation and Large Scale Structure*, Cambridge (2000).

Linde, A., *Particle Physics and Inflationary Cosmology*, Harwood (1990).

Ma, C., & Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gages*, *Astrophysical Journal* **455**, 7 (1995) [astro-ph/9506072].

Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, Freeman, 1973.

Mukhanov, V., Feldman, H., & Brandenberger, R., *Phys. Reports* 205, 203 (1992).

Padmanabhan, T., *Structure Formation in the Universe*, Cambridge (1993).

Peacock, J.A., *Cosmological Physics*, Cambridge (1999).

Peebles, P.J.E., *Principles of Physical Cosmology*, Princeton (1993).

Weinberg, S., *Gravitation and Cosmology*, Wiley (1973).

Black Holes (L24)

P. K. Townsend

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. The study of black holes reveals many surprising and beautiful properties, and has profound consequences for quantum theory. The following topics will be discussed:

1. Gravitational collapse. Why black holes necessarily form under certain circumstances.
2. Causal structure, asymptotic flatness, Penrose diagrams, the event horizon.
3. Exact black hole solutions: Schwarzschild, Reissner-Nordstrom and Kerr solutions.
4. Energy, angular momentum and charge. The positive energy theorem.
5. The laws of black hole mechanics. The analogy with laws of thermodynamics.
6. The Hawking effect. Black hole evaporation, the information paradox.

Examples sheets will be distributed during the course. Examples classes will be held to discuss these.

Desirable Previous Knowledge

Familiarity with the contents of the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Introductory Reading

1. R.M. Wald, *General Relativity* (University of Chicago Press, 1984), Chapter 6.

Reading to complement course material

1. P.K. Townsend, *Black holes: lecture notes*, arXiv:gr-qc/9707012.
2. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. *Spacetime and Geometry*, S.M. Carroll, Addison Wesley, 2004 (an earlier draft of this book is available online at pancake.uchicago.edu/~carroll/notes)
5. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.

Astrophysics

Galactic Astronomy and Dynamics (L24)

N.W. Evans

Astrophysics provides many examples of complex dynamical systems. This course covers the mathematical tools to describe Galaxies as well as reviewing their observational properties. The behaviour of these systems is controlled by Newton's laws of motion and Newton's law of gravity. Galaxies are dynamically very young, a typical star like the Sun having orbited only thirty or so times around the galaxy. The motions of stars in Galaxies are described using classical statistical mechanics, since the number of stars is so great. The study of large assemblies of stars interacting via long-range forces provides many unusual examples of cooperative phenomena, such as bars and spiral structure. The interplay between astrophysical dynamics and modern cosmology is also important – much of the evidence for dark matter is dynamical in origin

A detailed syllabus is as follows:

Observational overview. Stellar populations in galaxies, galaxy morphology and classification. Dust and gas in galaxies. Scaling Laws.

Theory of the gravitational potential. Poisson's equation. The gravity field of spherical, elliptical and disk galaxies. Regular and chaotic orbits, the epicyclic approximation, surfaces of section, action-angle coordinates, adiabatic invariance.

Collisionless stellar dynamics, the Boltzmann equation, the Jeans theorem, the Jeans equations, equilibrium models of spherical, elliptical and disk galaxies. Collisions, collisional dynamics, the Fokker-Planck equation. Globular cluster evolution, evaporation and ejection, the gravothermal catastrophe, hard and soft binaries.

Galactic stability, The Jeans length, theories of spiral structure, the role of resonances. The Milky Way Galaxy, the thin disk, thick disk and halo, substructure and tidal streams.

Desirable Previous Knowledge

The course is self-contained and suitable for astronomer and applied mathematicians. A knowledge of classical mechanics, methods of mathematical physics, and statistical physics would be helpful.

Introductory Reading

1. Sparke L., Galalgher J., Galaxies in the Universe: An Introduction, Cambridge University Press

Reading to complement course material

1. Binney, J., and Merrifield, M., Galactic Astronomy, Princeton University Press, 1999
2. Binney J., Tremaine S., Galactic Dynamics, Princeton University Press
3. Bertin G., Dynamics of Galaxies, CUP
4. Heggie D., Hut P., The Million Body Problem, CUP
5. Spitzer L., The Dynamical Evolution of the Globular Clusters, Princeton University Press

The Origin and Evolution of Galaxies (L24)

Martin Haehnelt

Galaxies are a fundamental building block of our Universe. The course will give an account of the physics of the formation of galaxies and their central supermassive black holes in the context of the standard paradigm for the formation of structure in the Universe.

Specific topics to be covered include the following:

- Observed properties of galaxies
- Cosmological framework and basic physical processes
- The interplay of galaxies and the intergalactic medium from which they form
- Numerical Methods for modeling galaxy formation
- Collapse of dark matter haloes and the inflow/outflow of baryons
- The hierarchical build-up of galaxies
- The origin and evolution of the central supermassive black holes in galaxies
- Towards understanding the origin of the Hubble sequence of galaxies

Desirable Previous Knowledge

The course is aimed at astronomers/astrophysicists (including beginning graduate students). It should be also suitable for interested physicists and applied mathematicians. The course is self-contained, but students who have previously attended introductory courses in General Relativity and/or Cosmology will have an easier start.

Introductory Reading

1. Ryden, B., *Introduction to Cosmology*, 2003, Addison-Wesley.
2. Sparke, L., Gallagher, J.S., *Galaxies in the Universe*, 2nd ed., 2007, Cambridge University Press.

Reading to complement course material

1. Mo, H., van den Bosch, F., White, S., *Galaxy Formation and Evolution*, 2010, Cambridge University Press.
2. Schneider, P., *Extragalactic Astronomy and Cosmology: An Introduction*, 2006, Springer.
3. Coles P., Lucchin F., *Cosmology - The Origin and Evolution of Cosmic Structure* (second edition), 2002, Wiley.

Dynamics of Astrophysical Discs (L16)

Gordon Ogilvie

A disc of matter in orbital motion around a massive central body is found in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary

star systems and in the centres of active galaxies, where they can reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

Occurrence of discs in various astronomical systems, basic physical and observational properties.

Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.

Evolution of an accretion disc.

Vertical disc structure, order-of-magnitude estimates and timescales, thin-disc approximations, thermal and viscous stability.

Shearing sheet, symmetries, shearing waves.

Incompressible dynamics: hydrodynamic stability, vortices.

Compressible dynamics (2D): density waves, gravitational instability.

Density waves in cylindrical geometry, Lindblad and corotation resonances.

Satellite-disc interaction, tidal potential, resonant torques, impulse approximation.

Magnetorotational instability.

Desirable previous knowledge

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is needed for the magnetorotational instability, but self-contained notes on this topic will be available.

Introductory reading

Much information on the astrophysical background is contained in

1. Frank, J., King, A. & Raine, D. (2002), *Accretion Power in Astrophysics*, 3rd edn, CUP.

Some of the basic theory of accretion discs is described in

1. Pringle, J. E. (1981), *Annu. Rev. Astron. Astrophys.* 19, 137.

Reading to complement course material

(There are no suitable textbooks.)

Astrophysical Fluid Dynamics. (M24)

John Papaloizou

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. While many fluid dynamical effects can be seen

in nature or the laboratory, there are other phenomena that are peculiar to astrophysics, for example self-gravitation, the dynamical influence of the magnetic field that is frozen in to a highly conducting plasma, and the dynamo effect driven by electromagnetic induction in a resistive fluid. The basic physical ideas introduced and applied in this course are those of Newtonian gas dynamics and magnetohydrodynamics (MHD) for a compressible fluid. The aim of the course is to provide familiarity with the basic phenomena and techniques that are of general relevance to astrophysics. Wherever possible the emphasis will be on simple examples, physical interpretation and application of the results in astrophysical contexts.

Examples of topics likely to be covered:

Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation. Microphysical basis and validity of a fluid description. Physical interpretation of MHD, with examples of basic phenomena, including dynamo theory. Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem. Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities. Spherically symmetric steady flows: stellar winds and accretion. Axisymmetric rotating magnetized flows: astrophysical jets. Waves and instabilities in stratified rotating astrophysical bodies.

Desirable Previous Knowledge

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.

Introductory Reading

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas*. Cambridge University Press.

Reading to complement course material

1. M.J.Thompson. *An Introduction to Astrophysical Fluid Dynamics* (2006). Imperial College Press.
2. Landau, L. D., and Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Pergamon Press.
3. Pringle, J. E., and King, A. R. (2007). *Astrophysical Flows*. Cambridge University Press.
4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: Gas Dynamics. University Science Books.

Structure and Evolution of Stars (M24)

C. A. Tout

The structure of a star can be mathematically described by certain differential equations which can be derived from the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics. Some familiarity with these theories will be assumed.

The basic equations of a spherical star will be derived in detail and the mode of energy transport, the equation of state, the physics of the opacity sources and the nuclear reactions will be discussed.

Approximate solutions of the equations for stellar structure will be given. Attention will be given to the virial theorem, polytropic gas spheres and homology principles. The procedure for numerical solution of the equations will be mentioned briefly.

The evolution of a star will be discussed with reference to its main-sequence evolution, the exhaustion of various nuclear fuels and the end points of evolution such as white dwarfs, neutron stars and black holes.

Close binary stars are two stars in a gravitationally bound system in which the ratio of orbital separation to stellar radius is sufficiently small for the stars to interact through tides and mass transfer. The concept of Roche lobe overflow and its stability will be discussed, as well as the effects on the stellar evolution. Algols and Cataclysmic Variables will be considered as observed examples.

Throughout the course, reference will be made to the observational properties of stars and these will be discussed at appropriate times with particular reference to the Hertzsprung–Russell diagram, the mass–luminosity law and spectroscopic information.

There will be four examples sheets each of which will be discussed during an examples class.

Desirable Previous Knowledge

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics though detailed knowledge of all of these is not expected.

Introductory Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Reading to complement course material

1. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
2. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.
3. Pringle, J. E. & Wade, R. A. *Interacting Binary Stars*, CUP, 1985.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of extrasolar planets (more than 500 are now known) and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems. Specific topics to be covered will be drawn from the following:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits
3. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
4. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments
5. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems

6. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap
7. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
8. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
9. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces

Desirable Previous Knowledge

This course is self-contained.

Reading to complement course material

1. Murray C. D. and Dermott S. F., *Solar System Dynamics*, Cambridge University Press, 1999
2. Armitage P. J., *Astrophysics of Planet Formation*, Cambridge University Press, 2010
3. Valtonen M. and Karttunen H., *The Three-Body Problem*, Cambridge University Press, 2006
4. Binney J. and Tremaine S., *Galactic Dynamics (Second Edition)*, Princeton University Press, 2008
5. de Pater I. and Lissauer J. J., *Planetary Sciences*, Cambridge University Press, 2010

Quantum Information and Quantum Foundations

Philosophy of Classical and Quantum Mechanics (M8 and L8)

Non-Examinable (Graduate Level)

Nazim Bouatta and Jeremy Butterfield

This graduate course analyses some philosophical aspects of classical and quantum physics. Since philosophy of physics is an inter-disciplinary subject (and the course is not examinable!), we propose to let the content be influenced by the interests of those attending. But we will begin the Michaelmas Term with elements of the quantum measurement problem (including density matrices, mixtures and decoherence); and then we will continue with recent work on quantization, and recent work on uncertainty relations (including an analogue in classical mechanics). In the Lent Term, we will discuss symmetry principles, including spacetime symmetries in classical field theories.

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the tools of quantum field theory, such as provided by the Part III courses, will be helpful but not essential.

Introductory Reading

This list of introductory reading is approximately in order of increasing difficulty.

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the frameworks of classical and quantum mechanics will be essential; but the technicalities of each topic will be developed as needed in the lectures.

Introductory Reading

1. Weyl, H. *Philosophy of Mathematics and Natural Science*. Princeton University Press.
2. Bell, J. *Speakable and Unspeakable in Quantum Mechanics*. CUP.

Reading to complement course material

1. Brading, K. and Castellani, E. (eds.) *Symmetries in Physics*. CUP.
2. Butterfield, J. On Symplectic Reduction in Classical Mechanics, in J. Earman and JNB (eds.) *The Handbook of Philosophy of Physics*, 2 volumes, Elsevier; pp. 1 - 131. Available at: [physics/0507194](http://philsci-archive.pitt.edu/archive/00002373/) and at <http://philsci-archive.pitt.edu/archive/00002373/>
3. Isham, C. *Modern Differential Geometry for Physicists*. World Scientific
4. Landsman, N. Between classical and quantum. In Butterfield, J. and Earman, J. (eds.) *Handbook of the Philosophy of Physics*, 2 volumes, Elsevier. Available at: <http://arxiv.org/abs/quant-ph/0506082> and at: <http://philsci-archive.pitt.edu/archive/00002328>
5. Sternberg, S. *Group Theory and Physics*. CUP.

Philosophical Foundations of Quantum Field Theory (M8 and L8)

Non-Examinable (Graduate Level)

Nazim Bouatta and Jeremy Butterfield

Quantum field theory (QFT) is a wonderful mountain range, combining strikingly deep and unifying ideas with a panoply of powerful calculational tools. In recent decades, QFT has become the framework for several basic and outstandingly successful physical theories. But it has been largely unexplored by philosophy of physics, which has concentrated on conceptual questions raised by non-relativistic quantum mechanics and general relativity (and the focus of another graduate course). Here, we will introduce the philosophical aspects of quantum field theory. More specifically, we will conceptually address topics that have been central to quantum field theory's development in the last forty years, such as: the renormalization group, spontaneous symmetry breaking, and anomalies.

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the tools of quantum field theory, such as provided by the Part III courses, will be helpful but not essential.

Introductory Reading

This list of introductory reading is approximately in order of increasing difficulty.

1. Wallace, D. (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Preprint available online at: <http://arxiv.org/pdf/quant-ph/0112148v1>
2. Weinberg, S. (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: <http://arxiv.org/abs/hep-th/9702027>
3. Douglas, M. (2011), 'Foundations of Quantum Field Theory'. Talk available online at: http://www.math.upenn.edu/StringMath2011/notes/Douglas_StringMath2011_talk.pdf
4. Fisher, M. (1998), 'Renormalization group theory: Its basis and formulation in statistical physics', *Rev. Mod. Phys* **70**, pp 653-681.

Reading to complement course material

1. Cao, T. ed. The Conceptual Foundations of Quantum Field Theory. Cambridge University Press, 1999.
2. Weinberg, S. The Quantum Theory of Fields, Vols I and II. Cambridge University Press, 1995 and 1996.
3. Ruetsche, L. Interpreting Quantum Mechanics. Oxford University Press, 2011.
4. Healey, R. Gauging What's Real. The Conceptual Foundations of Contemporary Gauge Theories. Oxford University Press, 2007.

Quantum Information Theory (L24)

Nilanjana Datta

Quantum Information Theory (QIT) is an exciting, young field which lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the

mathematical theory of acquisition, storage, transmission and processing of information. QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical analogues. These new features can be exploited, not only to improve the performance of certain information-processing tasks, but also to accomplish tasks which are impossible or intractable in the classical realm.

This is an introductory course on QIT, which should serve to pave the way for more advanced topics in this field. The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannons theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs and the Kraus Representation Theorem. Entanglement and some applications elucidating its usefulness as a resource in QIT will be discussed. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumachers theorem will be discussed in detail. The denition of ensemble average delity and entanglement delity will be introduced in this context. Various examples of quantum channels will be given and the dierent capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

Desirable Previous Knowledge

Knowledge of basic quantum mechanics will be assumed. However, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics. Elementary knowledge of Probability Theory, Vector Spaces and Group Theory will be useful.

Introductory Reading

The following book and lecture notes provide some interesting and relevant introductory reading material.

1. M.A.Nielsen and I.L.Chuang, Quantum Computation and Quantum Information; Cambridge University Press, 2000.
2. J.Preskill, Chapter 5 of his lecture notes: Lecture notes on Quantum Information Theory, <http://www.theory.caltech.edu/~preskill/ph229/#lecture>

Quantum Foundations (L16)

Berry Groisman

In recent decades, there has been a renaissance of interest in foundational issues in quantum theory, particularly in relation to quantum information science, cosmology and quantum gravity. This course provides an introduction to modern research on quantum foundations.

The first part of the course covers central cornerstones of quantum mechanics: quantum entanglement, the measurement problem, the Einstein-Podolsky-Rosen argument, Bell's theorem and quantum non-locality, experimental tests of quantum non-locality and the failure of local hidden variable theories, and the delicate "peaceful co-existence" between quantum theory and the no-signalling principle in special relativity. It is followed by detailed account of the theory of quantum measurement, including the von Neumann's measurement paradigm, state-verification measurements and generalized measurements. We then further develop the connection between quantum theory and relativity: we discuss the restrictions posed by special relativity on instantaneous measurements of non-local variables - properties of composite systems with space-like separated parts - and explore some recent developments in this subject.

In the second part of the course we review major foundational perspectives on quantum theory, starting with the "orthodox" standard Copenhagen theory, followed up by an early attempt at an alternative to

standard quantum theory, de Broglie-Bohm theory, and some of its problems. Then we go on to consider the physics of decoherence, some simple models of decoherence, and estimates of decoherence rates. This brings us to a more recent class of attempts at alternatives to quantum theory, the so-called 'dynamical collapse models' proposed by Ghirardi-Rimini-Weber, Pearle and others; we describe these models and review some of their problems. Finally, we discuss many-worlds quantum theory and the problem of making sense of probability in many-worlds theory. This in turn motivates a review of other approaches to the "quantum reality problem" - possible alternatives to quantum theory.

Desirable Previous Knowledge

A good understanding of undergraduate level quantum theory is required. (Cambridge 1B Quantum Mechanics course is a good starting point.)

Optional Introductory Reading

1. Benjamin Schumacher and Michael Westmoreland, *Quantum Processes Systems, and Information*, Cambridge University Press, Chapters 1-8. This is a good starter for those students who wish to review the core aspects of quantum theory in the context of quantum information. One might also find useful Benjamin Schumacher's lectures on Quantum Theory, video archived at <http://pirsa.org/C10028/>.

Optional reading to complement course material

1. John Bell, "Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press, 2nd edition, Chapters 1,2 and 22.
2. Robert Spekkens, lectures on Foundations of Quantum Mechanics given to Perimeter Scholars International (2009). Video archived at pirsa.org, beginning with the first lecture at <http://pirsa.org/09110168/>
3. Yakir Aharonov and Daniel Rohrlich, *Quantum Paradoxes: Quantum Theory for the Perplexed*, WILEY-VCH Verlag, Chapters 3, 7 and 14.
4. "Many Worlds? Everett, Quantum Theory, and Reality, Simon Saunders, Jonathan Barrett, Adrian Kent and David Wallace (eds.) (Oxford University Press, 2010), Chapter 8 (available at [arXiv:0906.2718](http://arXiv.org/abs/0906.2718)) and Chapter 10 (available at [arXiv:0905.0624](http://arXiv.org/abs/0905.0624)).

Quantum Computation (M16)

Richard Jozsa and Ashley Montanaro

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. Notable examples include integer factorisation (and consequent efficient breaking of commonly used public key crypto systems) and database searching. Apart from such practical benefits the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

The course will cover the following topics:

Notion of qubits, quantum logic gates, circuit model of quantum computation. Basic notions of quantum computational complexity, oracles, query complexity.

The quantum Fourier transform. Exposition of fundamental quantum algorithms including the Deutsch-Jozsa algorithm, Shor's factoring algorithm, the hidden subgroup problem and Grover's searching algorithm.

A selection from the following further topics:

- (i) Entanglement and the measurement-based model of quantum computation;

- (ii) Lower bounds on quantum query complexity;
- (iii) Applications of phase estimation in quantum algorithms;
- (iv) Quantum simulation;
- (v) Introduction to quantum walks.

Desirable Previous Knowledge

It is desirable to have familiarity with the basic formalism of quantum mechanics especially in the simple context of finite dimensional state spaces (state vectors, composite systems, unitary matrices, Born rule for quantum measurements). Revision notes will be provided giving a summary of the necessary material including an exercise sheet covering notations and relevant calculational techniques of linear algebra. It would be desirable for you to look through this material at (or slightly before) the start of the course. Any encounter with basic ideas of classical theoretical computer science (complexity theory) would be helpful but is not essential.

Reading to complement course material

1. Nielsen, M. and Chuang, I., Quantum Computation and Quantum Information. CUP.
2. John Preskill's lecture notes on quantum information theory, available at <http://www.theory.caltech.edu/~preskill/ph219/index.html/lecture>
3. Andrew Childs lecture notes on quantum algorithms available at <http://www.math.uwaterloo.ca/~amchilds/teaching/w11/qic823.html>

Applied and Computational Analysis

Distribution Theory and Applications (L16)

Dr A. Ashton

This course will provide an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will briefly look at Sobolev spaces in \mathbf{R}^n and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. We will also look at boundary value problems for elliptic PDEs in convex polygons. The course will be supplemented with hand-outs and example sheets. There will be three examples classes.

Desirable Previous Knowledge

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods/Analysis). No knowledge of measure theory or functional analysis is required.

Introductory Reading

1. F. G. Friedlander & M. S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.
3. G. B. Folland, *Introduction to Partial Differential Equations*, Princeton Univ Pr, 1995.

Reading to complement course material

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols I-II*, Springer Verlag, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics, Vols I-II*, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Numerical solution of differential equations (M24)

A. Iserles

The goal of this lecture course is to present and analyse efficient numerical methods for ordinary and partial differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural way to a wide range of numerical methods and computational strategies. The emphasis is on algorithms and their mathematical analysis, rather than on applications.

The course consists of three parts: methods for *ordinary differential equations* (with an emphasis on initial-value problems and a thorough treatment of stability issues and stiff equations), numerical schemes for *partial differential equations* (both boundary and initial-boundary value problems, featuring finite differences and finite elements) and, time allowing, *numerical algebra of sparse systems* (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods). We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms.

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics *per se*, except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will neither be assumed nor is necessarily an advantage. Experience with programming and application of computational techniques will obviously aid comprehension but is neither assumed nor expected.

Fluency in linear algebra and decent understanding of mathematical analysis are a must. Thus, linear spaces (inner products, norms, basic theory of function spaces and differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following present material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you to acquire valuable background in numerical techniques:

Introductory Reading

1. S. Conte & C. de Boor, *Elementary Numerical Analysis*, McGraw–Hill, New York, 1980.
2. G.H. Golub & C.F. van Loan, *Matrix Computations*, 3rd edition. Johns Hopkins Press 1996.
3. M.J.D. Powell, *Approximation Theory and Methods*, Cambridge University Press, Cambridge, 1981.
4. G. Strang, *Introduction to Linear Algebra*, Wellesley-Cambridge Press, Cambridge (Mass.), 3rd ed. 2003..

Reading to complement course material

1. U. M. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, Philadelphia, 2008.
2. O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1996.
3. E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, Berlin, 2nd ed. 1993.
4. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems*, Springer-Verlag, Berlin, 2nd ed. 1996.

5. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, Cambridge, 2nd ed. 2008.
6. K.W. Morton & D.F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction*, Cambridge University Press, Cambridge, 2005.
7. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Cambridge (Mass.), 2nd ed. 2008.

Approximation Theory (M24)

A Shadrin

The course will give an overview of basic concepts of Approximation Theory, i.e., best and good approximation of a large family of functions by a smaller set (usually finitely generated, linear or nonlinear) in certain normed spaces (such as C and L_p), construction of good approximants (if possible), finding approximation order.

We start with the classical polynomial tools and then move the emphasis to univariate splines. Time allowing, we will pay some attention to multivariate splines and wavelets.

Desirable Previous Knowledge

Some elements of Functional Analysis (normed spaces, linear operators).

Reading to complement course material

1. E. W. Cheney, *Approximation theory*, McGraw-Hill, New-York, 1966.
2. R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.
3. C. de Boor, *Lecture Notes on Approximation Theory*, www.cs.wisc.edu/~deboor

Reaction-Diffusion Equations. (M16)

M-T. Wolfram

The course presents reaction-diffusion equations both as partial differential equations featuring highly interesting mathematical phenomena as well as mathematical models of prominent applications in biology, life-sciences, chemistry, population dynamics, ...

The aim is to introduce the fundamental ideas of the existence theory of solutions and to describe the techniques which analyse the qualitative and the large-time behaviour of solutions in terms of selected model applications.

Full lecture notes will be available.

Outline: 1) Introduction: some comments on - the mathematical modelling with continuum equations - famous reaction-diffusion models (e.g. Hodgkin-Huxley, Belouso-Zhabotinsky, ...) - parabolic partial differential equations (e.g. comparison principles)

2) Existence theory of reaction-diffusion equations (what is a "good" solution and how to construct it): - local existence: fixed point argument, regularity bootstrap, well-posedness - global existence: a priori estimates,

3) Qualitative and large time analysis: - stationary states and their stability, - Lyapunov functionals, Entropy methods

4) Patterns and waves: - travelling waves, Fisher's equation, stability of travelling waves - Turing instability - pattern formation, activator-inhibitor models

Desirable Previous Knowledge

Some basic knowledge of measure theory (Lebesgue integrals), functional analysis (Lebesgue spaces) and ordinary differential equations would be desirable but will not be necessary.

Introductory Reading

1. Fife, P. C. *Mathematical Aspects of Reacting and Diffusing Systems*. Lecture Notes in Biomathematics 28. Springer (1979).
2. Grindrod, P. *Patterns and Waves: The Theory and Applications of Reaction-Diffusion Equations*. Clarendon Press (1991).
3. Murray, J. D. *Mathematical Biology*. Springer (2002).
4. Rothe, F. *Global Solutions of Reaction-Diffusion Systems*. Lecture Notes in Mathematics 1072. Springer (1984).
5. Smoller, J. *Shock Waves and Reaction Diffusion Equations*. Springer (1994).

Reading to complement course material

1. Evans, L. C. *Partial Differential Equations*. Graduate Studies in Mathematics. AMS (1998).
2. Meinhardt, H. *The Algorithmic Beauty of Sea Shells*. Springer (2003).
3. Pao, C. V. *Nonlinear Parabolic and Elliptic Equations*. Springer (1993).

Continuum Mechanics

Fundamentals in Fluid Dynamics of Climate (M24)

S.B. Dalziel, J.R. Taylor, J.A. Neufeld and P.F. Linden

Understanding and predicting the Earth's climate is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the Earth's climate, helping to make the planet hospitable for life. However, the dynamical complexity of this motion and the wide range of space and time scales involved, makes predicting the climate system a very difficult endeavour.

This course provides an introduction to the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focusing on the behaviour of stratified and rotating flows. The course begins by considering systematic flows driven by, or inhibited by, density differences. Internal gravity waves, where the density variations *within* the fluid provide the restoring force, radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input. Density interfaces also support the propagation of waves, but in many geophysical and environmental contexts the system may be considered 'shallow', leading to substantial simplifications. The second main section of the course deals with flows where the timescale for the motion is not short compared with a day and the Earth's rotation plays an important role. The additional timescale introduced by the Earth's rotation modifies the dynamics in a profound way for both homogeneous and density stratified flows. The Coriolis force (a fictitious force arising from our use of a frame of reference rotating with the planet) causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics. The course concludes with an introduction to turbulence, exploring some of its basic characteristics and how its presence can require additional terms to be introduced into the simplified equations derived for stratified and rotating flows.

Although not a prerequisite in either case, aspects of the material covered in this course are developed further in Advanced Topics in Fluid Dynamics of Climate (L24) and in Turbulence (L16).

Desirable Previous Knowledge

Undergraduate fluid dynamics

Reading to complement course material

1. A.E. Gill, Atmosphere-Ocean Dynamics. Academic Press (1982).
2. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. 2008.
3. Pedlosky, J. Geophysical Fluid Dynamics. Springer. (1987).
4. J.S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Turbulence (L16)

P A Davidson

Course Description

This course provides an introduction to turbulence, with an emphasis on fundamental concepts. The following will be covered.

1. *Review of Vortex Dynamics* [2L]
 - Vorticity and angular momentum. The vorticity equation: advection, diffusion and stretching of vorticity. Burgers' vortex. Inviscid flow: Helmholtz's laws and Kelvin's theorem.
 - The Biot-Savart law. Linear and angular impulse and far-field properties of vortex blobs. Turbulence as a chaotic vorticity field.
2. *Transition to Turbulence* [1L]

Non-linearity and chaos. Logistic equation as an example. Transition to turbulence.
3. *Elementary Properties of Homogeneous Turbulence* [1L]

The wide range of scales in turbulence. A glimpse at Richardson's energy cascade. The Kolmogorov scales in fully-developed turbulence. The zero law of turbulence. Sensitivity to initial conditions and the need for a statistical approach. Order within chaos: turbulence has a memory. Two-point correlations and the closure problem. Properties of the auto-correlation function. The pdf of the velocity field and its derivatives. Flatness and skewness factors.
4. *Statistical Measures of the Distribution of Energy Across the Scales* [2L]
 - The need for a filter which gives a scale-by-scale measure of energy. The spectral approach: the spectral tensor and the energy spectrum. A real-space approach: structure functions.
 - Relating real-space and spectral-space filters. Problems with these statistical measures: the loss of phase information, tails in the distributions.
5. *The Energy Cascade, Cascades in Passive Scalar Mixing, Intermittency* [3L]
 - Richardson's cascade and Kolmogorov's theory of the small scales (K41). The 2/3rd and 4/5th laws. Do we ever observe Kolmogorov turbulence at terrestrial values of the Reynolds number?: the persistence of anisotropy, the failure of the 4/5th law, the 5/3rd law and the turbulent bottle neck.
 - Cascades in passive scalar mixing and the effect of Schmidt number.
 - Intermittency: evidence that the small scales are spotty. Landau's objection to Kolmogorov's theory. Anomalous scaling in higher-order structure functions. Kolmogorov's refined theory (K62) and other refined models. How reliable is the experimental evidence for high-order structure functions.
6. *Vortex line stretching* [1L]

Enstrophy production by vortex-line stretching. Are the vortices tubular or sheet like? Betchov's theory. The cascade is not self-similar. The taxonomy of worms.
7. *Turbulent diffusion* [1L]

Taylor diffusion for a single particle. Richardson's law for the relative diffusion of two particles.
8. *Decaying isotropic turbulence* [3L]
 - The Karman-Howarth equation and the closure problem. The 4/5th law (again). The skewness factor and enstrophy production. Simple closure models.
 - k^2 versus k^4 spectra and their properties. Statistical invariants and their relationship to momentum conservation. The importance of initial conditions. The classical theory of decay for k^4 spectra: Kolmogorov's decay laws.
 - Long-range pressure forces and the objections of Batchelor to Kolmogorov's decay laws. The Saffman-Birkhoff decay law for k^2 spectra. The evidence from numerical simulations.
9. *Turbulent shear flows* [2L]
 - Reynolds equation and Reynolds stresses. The closure problem (yet again!). Jets and wakes.
 - Turbulence near a wall: the log-law for the mean velocity, the k^{-1} law for the energy spectrum, the log-law for temperature, coherent structures and the near wall cycle.

Desirable Previous Knowledge

It is highly recommended that students have taken the Part II fluids course (or its equivalent) as some knowledge of vortex dynamics will be assumed. It is also assumed that students are conversant with vector calculus and Fourier transform theory.

Introductory Reading

1. Acheson, D. J., *Elementary Fluid Mechanics*, Clarendon Press, 1990. [Chapter 5 is ideal for those who want to brush up on their vortex dynamics.]
2. Davidson, P. A., *Turbulence, An Introduction for Scientists and Engineers*, OUP, 2004. [Chapter 1 provides a simple introduction to turbulence, chapter 2 covers those aspects of fluid mechanics with which the student is assumed to be familiar, and §8.1.1-§8.1.3 summarizes those results from Fourier analysis which are required for the course.]

Reading to Complement Course Material

1. Batchelor, G. K., *The Theory of Homogeneous Turbulence*, CUP, 1953. [The book that defined the subject. The style is somewhat terse.]
2. Davidson, P. A. *Turbulence, An Introduction for Scientists and Engineers*, OUP, 2004. [The course more or less follows the first half of this text.]
3. Drazin P. G., *Nonlinear Systems*, CUP, 1992. [The transition to chaos and the logistic equation are covered in chapter 3.]
4. Townsend, A. A., *The Structure of Turbulent Shear Flow*, CUP, 1976. [Broader in outlook than the title suggests.]

Examples Classes and Examples Sheets

There will be a total of 6 examples sheets and 8 examples classes.

Biological Physics (M24)

R.E. Goldstein & U. Keyser

This course will provide an overview of the physics and mathematical description of living systems. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology. The topics to be covered will span the range of length scales from molecular to ecological, with emphasis on key paradigms. Introductory material on statistical mechanics will provide background for much of the course. The subsequent topics will include *Microscopic Physics* – van der Waals forces, screened electrostatics, Brownian motion, fluctuation-dissipation theorem; *Fluctuation-Induced Forces* – polymer physics, random walks, entropic forces, stiff chains, self-avoidance, dynamics, protein folding; *Elasticity* – differential geometry of curves and surfaces, linear elasticity theory, thin plates and rods, Helfrich model for membranes, elastohydrodynamics; *Chemical Kinetics and Pattern Formation*– Michaelis-Menten kinetics, oscillations, excitable media, ion channels, action potentials, reaction-diffusion dynamics, Fitzhugh-Nagumo model, spiral waves; *Dynamics*– life at low Reynolds numbers, chemoreception, advection-diffusion problems.

Desirable Previous Knowledge

Some familiarity with statistical physics will be helpful.

Introductory Reading

1. P. Nelson. *Biological Physics*. W.H. Freeman (2007).
2. J.D. Murray. *Mathematical Biology I. & II*. Springer (2007, 2008).
3. K. Dill & S. Bromberg. *Molecular Driving Forces*. Garland (2009).

Reading to complement course material

1. B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts and P. Walter. *Molecular Biology of the Cell*. 5th edition. Garland Science (2007).
2. J.N. Israelachvili. *Intermolecular and Surface Forces*. 2nd edition. Academic Press (1992).
3. E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids*. Elsevier (1948).
4. M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics*. OUP (1986).
5. A. Parsegian. *Van der Waals Forces*. CUP (2005).
6. D. Andelman & W. Poon. *Condensed Matter Physics in Molecular and Cell Biology*. Taylor & Francis (2006).

Computational Methods in Fluid Mechanics (M16)

Non-Examinable (Graduate Level)

E.J. Hinch

The aim of this course is to provide an overview of some of the computational methods used to solve the partial differential equations that arise in fluid dynamics and related fields. The idea is to provide a feel for the computational methods rather than study them in depth (cf. the complementary aim of the course on the Numerical Analysis of Differential Equations). Although the course is non-examinable, project-type essays will be set on some of the material.

The course will start with a four-lecture introduction to the numerical solution of the Navier-Stokes equations at moderate Reynolds number; the issues and difficulties will be highlighted.

Next some general issues will be covered in greater detail.

- Discretisations: finite difference, finite element and spectral.
- Time-Stepping: explicit, implicit, multi-step, splitting, symplectic.
- Solution of Linear Systems: dense solvers, structured matrices, iterative methods: multigrid, conjugate gradient, GMRES and alternatives, preconditioning, sparse direct methods, eigensolvers, pseudo-spectra.

The remaining lectures will focus on specific issues selected from the following.

- Demonstration of the commercial software FLUENT.
- Implementation issues: code design, testing, data prefetch, cache issues, use of black-box routines, modular code design, language compliance.
- Methods for hyperbolic systems of equations such as the compressible Euler equations.
- Representation of surfaces: splines for curves, diffuse interface method, indicator functions in Volume of Fluid methods, level sets.
- Boundary Integral/Element Method.
- Fast Multipole Method.
- Parameter continuation.
- Lattice-Boltzmann and similar methods.

Pre-requisite Mathematics

Attendance at an introductory course in Numerical Analysis that has covered (at an elementary level) the solution of ordinary differential equations and linear systems will be assumed. Some familiarity with the Navier-Stokes equations and basic fluid phenomena will be helpful (as covered by a first course in Fluid Dynamics).

Literature

1. Boyd, J.P. (2000) *Chebyshev and Fourier Spectral Methods* Dover.
2. Acton, F.S. (1990) *Numerical Methods That Work* Mathematical Association of America.
3. LeVeque, R.J. (1992) *Numerical Methods for Conservation Laws* Birkhauser-Verlag.
4. Saad, Y. (1996) *Iterative Methods for Sparse Linear Systems* PWS.
5. Barrett, R. et al. (1994) *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods* SIAM.
6. Iserles, A. (1996) *A First Course in the Numerical Analysis of Differential Equations* CUP.

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media will be analysed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Desirable Previous Knowledge

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Introductory Reading

1. D.J. Acheson. *Elementary Fluid Dynamics*. OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics*. CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes*. Butterworth (1992). Chapters 4 and 5.

Reading to complement course material

1. J. Happel and H. Brenner. *Low Reynolds Number Hydrodynamics*. Kluwer (1965).
2. S. Kim and J. Karrila. *Microhydrodynamics: Principles and Selected Applications*. (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow*. CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks*. CUP (1991).

Perturbation and Stability Methods (M24)

J.M. Rallison and N. Peake

This first part of this course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of the most useful mathematical tools for research will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called ‘exponential asymptotics’), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

The second part of the course covers applications of perturbation methods to the study of fluid flows. So-called ‘hydrodynamic stability’ is a very broad discipline, and in this course we will concentrate on the stability of nearly parallel-flows (as for example arise in boundary-layer flows).

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals*. This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the ‘divide and conquer’ strategy, Laplace’s method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to ‘asymptotics beyond all orders’ in which exponentially small corrections are extracted from the tails of asymptotic series. [6]
- *Multiple Scales*. This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the ‘WKBJLG’ method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of ‘averaging’ and ‘time scale distortion’ in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium, including investigation of the rescaling required near ‘hot spots’, or ‘caustics’). [5]
- *The Summation of Series*. Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks’ transformations, Richardson extrapolation, Domb-Sykes plots. [1]
- *Matched Asymptotic Expansions*. This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. At the end of this section further examples will be given of asymptotics beyond all orders. [6]
- *Stability Theory*. This section will review both eigenvalue and ‘non-eigenvalue’ aspects of stability theory as applied to fluid flows, concentrating on nearly-parallel flows. Aspects that will be covered include the concepts of ‘causality’ and the Briggs-Bers technique, the continuous spectrum, and the transitory algebraic growth that can follow from the fact that the operators in hydrodynamic stability theory are often not self-adjoint. [6]

In addition to the lectures, a series of examples sheets will be provided. The lecturers will run examples classes in parallel to the course.

Desirable Previous Knowledge

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve simple differential equations and partial differential equations and evaluate simple integrals.

Introductory Reading

1. E.J. Hinch. *Perturbation Methods*, Cambridge University Press (1991).
2. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975).

Reading to complement course material

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *Beware: Bender and Orszag call Stokes lines anti-Stokes lines, and vice versa.*
2. John P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series* Acta Applicandae, **56**, 1-98 (1999), and also available at

<http://www-personal.engin.umich.edu/~jpboyd/boydactaapplicreview.pdf>

3. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265-280 (1990).
4. P.G. Drazin and W. H. Reid. *Hydrodynamic Stability*, Cambridge University Press (1981 and 2004).
5. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer (1981).
6. Peter Schmid and Dan S. Henningson. *Stability and Transition in Shear Flows*, Springer-Verlag (2001).

Convection (L24)

Prof. M.R.E.Proctor

Convection is the name given to the means used by fluids to transfer heat when fluid flow is more effective than conduction. In a fluid layer, for example, between horizontal boundaries held at fixed temperatures, convection occurs when the temperature difference is sufficiently large. The onset of convection can be thought of as an instability of pattern forming type, and there are many interesting questions that can be asked: What are the pattern and horizontal scale of convection near onset? How does the heat transfer depend on the temperature difference? How do the simple patterns seen at onset break down? What is the effect on convection of other physical effects such as rotation, and what happens when there are two sources of buoyancy, such as thermohaline convection?

The course will address many of these issues. Though convection requires that fluid density depend on temperature and so be non-uniform, most of the course will use the Boussinesq approximation, in which the fluid may be treated as incompressible except for the buoyancy term. This approximation is a good one for laboratory liquids and gives a good guide to many aspect of convection for which the approximation is not accurate.

There will be three problem sheets and associated examples classes.

Desirable Previous Knowledge

Knowledge of fluid dynamics and dynamical systems would be an advantage.

Introductory Reading

1. Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability. Dover
2. Drazin, P and Read, W. Hydromagnetic stability (chapter 2). CUP

Reading to complement course material

1. Getling, A.V. Rayleigh-Benard convection: structures and dynamics. World Scientific
2. Hoyle, R. Pattern Formation. CUP

Wave Propagation and Scattering. (L24)

Orsola Rath-Spivack and Ed Brambley

The application of wave theory to problems involving the generation, propagation and scattering of acoustic and other waves is of considerable relevance to many applications of practical significance. These include, for example, underwater sound propagation, aircraft noise, remote sensing, the effect of noise in built-up areas, and a variety of medical diagnostic applications.

This course aims to provide the basic theory of wave generation and scattering, and an overview of the mathematical methods and approximations used to tackle these problems, with emphasis on applications to aeroacoustics, propagation and scattering in random media, and inverse scattering.

The first part of this course will cover some general aeroacoustic theory (including Lighthill's theory of noise generation) [6], long-distance sound propagation (including nonlinear and viscous effects) [2], wave-guides [2], and application of the scalar Wiener-Hopf technique to scattering of a plane wave by a sharp edge [2].

The second part of this course will be concerned mainly with acoustic wave propagation and scattering in inhomogeneous media and with the inverse scattering problem. The parabolic equation and Born and Rytov approximations for the scattering problem will be first introduced [2]; scattering by randomly rough surfaces and propagation in inhomogeneous media will then be described in some detail using perturbation methods and other approximations to derive equations and solutions for the mean field and higher moments [5]; finally, Tikhonov regularization and other techniques for the regularization of inverse problems will be considered, and some methods for the solution of the inverse scattering problem will be introduced [5].

In addition to the lectures, several printed handouts and some lecture notes will be provided. The lectures will be supplemented by example sheets and example classes

Desirable Previous Knowledge

This course assumes that students have attended some introductory courses in Continuum Mechanics and Complex Variable Theory (especially Fourier Transforms and the use of complex residues). Some familiarity with the wave equation and with linear algebra is helpful. Attendance at the Part III course on Perturbation Methods would also be helpful for a portion of the first half of the course, but is by no means essential. Some concepts in the second half of the course complement the Part III course on Image Processing and Variational Methods.

Introductory Reading

For example:

1. Dowling, AP and Ffowcs Williams, JE. *Sound and Sources of Sound*, Ellis Horwood.
2. Landau, LD and Lifschitz, EM. *Fluid Mechanics*, Butterworth-Heinemann. [Chapter 8]
3. Groetsch, CW *Inverse Problems in the Mathematical Sciences*, Braunschweig 1993

Reading to complement course material

1. Crighton, DG et al. *Modern Methods in Analytical Acoustics*, ASA 1989.
2. Jones, JD. *Acoustic and Electromagnetic Waves* Clarendon Press 1986
3. Ishimaru, A *Wave Propagation and Scattering in Random Media*, Academic Press, 1978.
4. Colton, D and Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory* ,Springer Verlag 1992.

Advanced Topics in Fluid Dynamics of Climate. (L24)

J.R. Taylor, S. Dalziel, P.F. Linden and J.A. Neufeld

Understanding and predicting the Earth's climate is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the Earth's climate, helping to make the planet hospitable for life. However, the dynamical complexity of this motion and the wide range of space and time scales involved, makes predicting the climate system a very difficult endeavour. This course is designed to follow Fundamentals in Fluid Dynamics of Climate (M24) building upon knowledge of rotating and stratified fluid mechanics to describe the dynamics of the ocean and atmosphere. The aim of the course is to survey advanced topics and explore open questions relevant to the climate system, while providing a strong background for research in geophysical fluid dynamics.

The effects of fluid density and the Earth's rotation depend strongly on the spatial scale of the process considered. To reflect this, the course will be arranged into three sections, each examining a different dynamical regime. The first section will examine large-scale dynamics where the rotation of the Earth is a dominant control, including the phenomena commonly referred to as Rossby waves, eddies, baroclinic instability, ocean gyres and the thermohaline circulation. These processes play a central role in the lateral transport of heat and other tracers. The second section will describe smaller-scale processes where rotation has a weaker influence but density effects are still important, including stratified turbulence, boundary layers, ocean mixing and circulation in the deep ocean. These processes are instrumental in setting the vertical structure of the ocean and atmosphere. While the first two sections focus on cases when the fluid density has generally a stabilizing effect, the third section will describe buoyancy-driven flows where the fluid density is destabilizing and drives the fluid motion. Topics will include gravity currents, jets, plumes and thermals, and natural ventilation.

Desirable Previous Knowledge

A basic understanding of fluid mechanics (undergraduate level) is required. Fundamentals of the Fluid Dynamics of Climate Change (M24) is strongly recommended, but not required for the Advanced section. The student should start with a firm grasp of stratified and rotating fluids.

Introductory Reading

1. Turner, J.S. *Buoyancy Effects in Fluids*. CUP. 1973.
2. Marshall, J. and R.A. Plumb. *Atmosphere, Ocean, and Climate Dynamics*. Academic Press. 2008.

Reading to complement course material

1. Gill, A.E. *Atmosphere-Ocean Dynamics*. Academic Press. 1982.
2. Pedlosky, J. *Geophysical Fluid Dynamics*. Springer. 1987.

The Physics Of The Polar Oceans, Sea Ice and Climate Change (L16)

Peter Wadhams

Course description

The course is designed to give a complete background on the physics of sea ice and its role in the climate system, also including ice mechanics, icebergs and the physics of oil-ice interaction. The course comprises 16 sessions, as shown below. Each session requires an hour of teaching.

1. Regional setting The geography, water structure, currents and ice cover in the Arctic and Antarctic oceans.
2. The physics of sea ice and ice formation What happens when sea water cools Growth of ice crystals Brine cells and brine rejection Salinity structure Summer melt processes First- and multi-year ice
3. Ice growth and decay Thermodynamic model Equilibrium thickness Sensitivity of thickness to changes in forcing Sensitivity to albedo.
4. Ice dynamics Ice motion - driving forces Free drift solution Ice interaction The dynamics of polynyas
5. The ice thickness distribution Ridge and lead formation Geometry of pressure ridges The probability density of ice thickness and its evolution Mathematical form of ridges and leads distributions
6. Ice mechanics The ridging and rafting process Ridge evolution and decay Ice interaction with structures Ice interaction with the seabed
7. The marginal ice zone Ice floes Waves in ice Modelling development of floe size distribution Eddies
8. Icebergs and ice islands Sources Distribution in Arctic and Antarctic Physical properties Dynamics Decay and breakup Role in the oceans and in sediment transport
9. Glacial ice threat to offshore structures Iceberg scouring depths, incidence, seabed interaction Mechanics of iceberg and ice island interaction with structures Upstream detection of ice islands
10. Oil spills under ice Scope of the under ice blowout problem Other sources of spills under and in ice Physical behaviour of crude oil in very cold water Dynamics of a rising oil-infested bubble plume Incorporation of oil in rough sea ice containment factors Ice growth under an oil layer Oil penetration into brine drainage channels Oil transport by ice The melt process and mode of final oil release Oil behaviour in pancake ice and the marginal ice zone
11. Regional ice studies Greenland and the Beaufort Sea East Greenland waters Greenland Sea convection zone South Greenland and the Storöf Baffin Bay ice conditions Nares Strait The Lincoln Sea and waters north of Greenland The Beaufort Gyre and its variability Changes in ice conditions in central Beaufort Sea The Beaufort Sea coastal zone The summer Beaufort Sea as a new MIZ Methane release from seabed
12. Thinning and retreat of sea ice Satellite data on retreat Parkinson - retreat in sectors, Arctic and Antarctic What is found in Antarctic Thinning - the submarine and other evidence Model predictions of a future seasonal Arctic ice cover
13. Conclusions - how did it start and where will it end? Ice ages and their causes Earlier ice-free periods The physics of the greenhouse effect Is Man the only cause of current changes? What will happen in the longer term? Global sea level rise Potential feedbacks on feedbacks.
- 14-16. Special topics and examples, drawn from rest of course and inserted at appropriate intervals (probably after 5, 9 and 13).

Desirable previous knowledge

Basic fluid dynamics and some knowledge of, and interest in, climatology, oceanography and glaciology would be useful.

Introductory reading

Background reading of general introductory physical oceanographic books will be useful. The book of the course is

"Ice in the Ocean" by P Wadhams (Taylor and Francis, 2000).
This might be read through before the lectures begin.

Reading to complement course

Other very useful books which will be used in the course are

"Global Warming - the Complete Briefing" by Sir John Houghton, 3rd Edn (CUP)

and

"On Sea Ice" by Willy Weeks (Univ. Alaska Press).

During the course there will be specific references to material that should be pursued further in sources such as

The Geophysics of Sea Ice (ed. N Untersteiner)

The Physics of Ice-Covered Seas (Univ Helsinki)

The Drift of Sea Ice (M Lepparanta)

Field Techniques for Sea Ice Research (ed. H. Eicken)