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Where you worked: Cambridge

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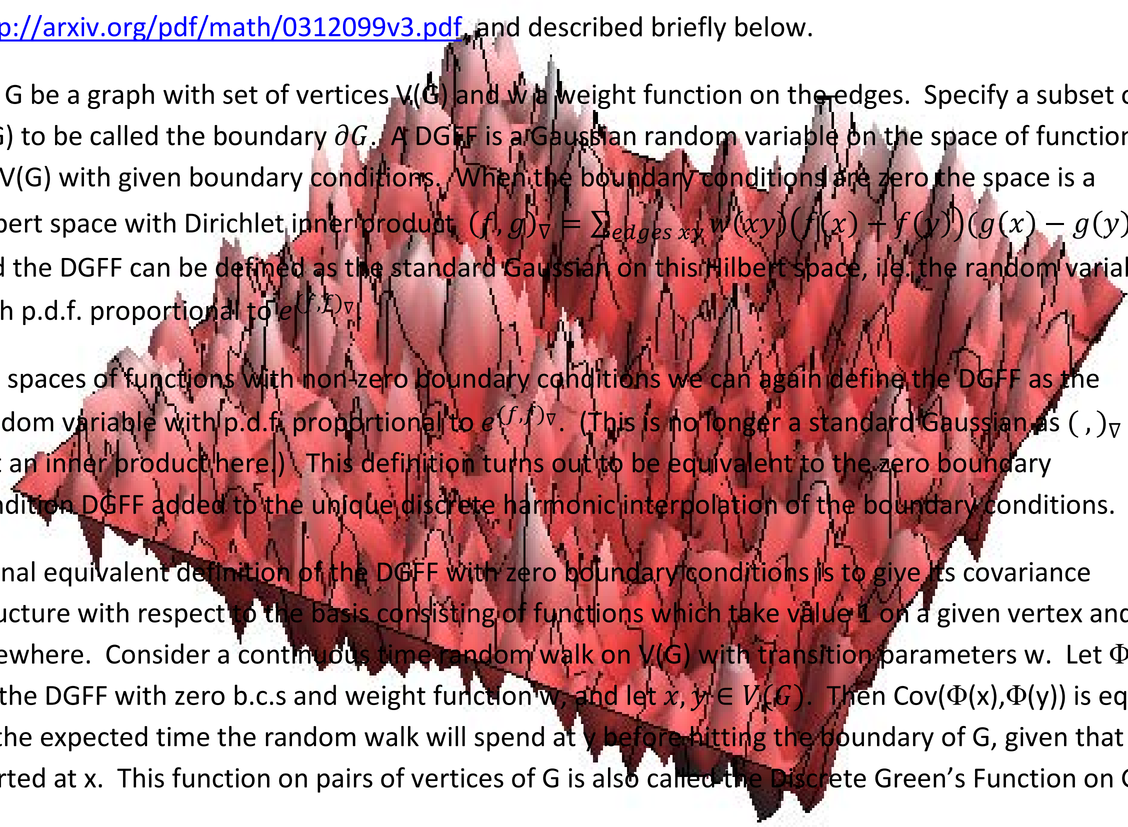
Subject: Discrete Gaussian Free Fields

Partial Summary:

My project involved proving the basic facts about the Discrete Gaussian Free Field (DGFF), as stated in Scott Sheffield's "Gaussian Free Fields for Mathematicians", http://arxiv.org/pdf/math/0312099v3.pdf, and described briefly below.



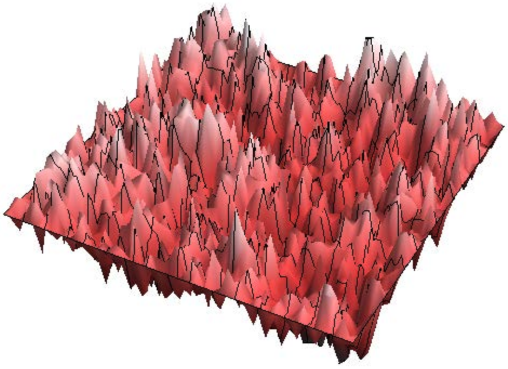
Let G be a graph with set of vertices V(G) and w a weight function on the edges. Specify a subset of V(G) to be called the boundary ∂G. A DGFF is a Gaussian random variable on the space of functions on V(G) with given boundary conditions. When the boundary conditions are zero the space is a Hilbert space with Dirichlet inner product $(f,g)_\nabla = \sum_{edges\ xy} w(xy)\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)$, and the DGFF can be defined as the standard Gaussian on this Hilbert space, i.e. the random variable with p.d.f. proportional to $e^{(f,f)_\nabla}$.

For spaces of functions with non-zero boundary conditions we can again define the DGFF as the random variable with p.d.f. proportional to $e^{(f,f)_\nabla}$. (This is no longer a standard Gaussian as $(\ ,\ )_\nabla$ is not an inner product here.) This definition turns out to be equivalent to the zero boundary condition DGFF added to the unique discrete harmonic interpolation of the boundary conditions.

A final equivalent definition of the DGFF with zero boundary conditions is to give its covariance structure with respect to the basis consisting of functions which take value 1 on a given vertex and 0 elsewhere. Consider a continuous time random walk on V(G) with transition parameters w. Let Φ be the DGFF with zero b.c.s and weight function w, and let $x, y \in V(G)$. Then Cov(Φ(x),Φ(y)) is equal to the expected time the random walk will spend at y before hitting the boundary of G, given that it started at x. This function on pairs of vertices of G is also called the Discrete Green's Function on G.

The DGFF has a domain Markov property: Let H be a subgraph of G, and give it boundary (∂H) consisting of those vertices which are neighbours to vertices of G\H. Let Φ be a DGFF on G. Conditional on the values of $\Phi|_{G\setminus H \cup \partial H}$, $\Phi|_H$ is the DGFF on H with boundary conditions $\Phi|_{\partial H}$.

Given a compact domain $D \subseteq \mathbb{R}^n$, let H(D) be the completion of the space of compactly supported functions on D. This is a Hilbert Space with a continuous Dirichlet inner product. The Gaussian Free Field (GFF) is a random linear functional $H(D) \to \mathbb{R}$ which is a generalisation of "standard Gaussian random variable" to this space.   Let T be a tessellation of $\mathbb{R}^n$ by triangles in which D meets finitely many triangles and let H(T) be the subspace of H(D) of functions affine on all triangles of T. This is a finite dimensional Hilbert space so restriction of the action of a GFF sample to H(T) gives an element of the dual space to H(T) which corresponds to an element of H(T). It turns out that this random variable on H(T) is the DGFF with weight function w, where w(e) is half the sum of the cotangents of the two angles opposite e.



Sample of the Discrete Gaussian Free Field with zero b.c. on a 60x60 square lattice (image from Wikipedia)