Formalising Algebraic Geometry

$$\dim_k H^p(X, \mathcal{F}) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{F}^\vee)$$

```
Sheaves.thy (%USERPROFILE%\Documents\PMP Project\)
lemma (in presheaf) stalkrel_is_equiv:
fixes x
shows  "equiv (prestalk x) (stalkrel x)"
proof -
 have refl: "refl_on (prestalk x) (stalkrel x)"
unfolding refl_on_def using stalkrel_def prestalk_def presheaf_axioms stalkrel_refl  by auto
then have sym: "sym (stalkrel x)"
unfolding sym_def
using stalkrel_def prestalk_def presheaf_axioms stalkrel_sym by auto
then have trans: "trans (stalkrel x)"
unfolding trans_def
using stalkrel_def prestalk_def presheaf_axioms stalkrel_trans
by (smt ‹refl_on (prestalk x) (stalkrel x)› case_prodE mem_Collect_eq refl_onD1 refl_onD2)
then show ?thesis
using refl trans sym by (simp add: equiv_def)
qed
```

Figure 1: Isabelle proof that the germ relation is an equivalence relation

Isabelle

In short, Isabelle is a proof assistant software that:

- Allows the development of mathematical theories;
- With the aid of an user, make formal verification of facts (theorems);

This makes an Isabelle formalisation much more than simply a program that emulates mathematics - it uses logic and (with the aid of an user) previously verified facts to check the correction of what's been done. It is expected that Isabelle and other systems such as Coq will play an increasing role in mathematical research, with prominent mathematicians such as Fields medalists Vladimir Voevodsky (IAS - Princeton) and Timothy Gowers (Cambridge) demonstrating an interest in computer verification and (semi-)automatic theorem proving.
Isabelle in particular has a very useful automatic component in the form of *sledgehammer*, a tool that allows the software to attempt to complete a proof by itself, resorting to existing information: if proofs are found, they are suggested to the user.
The structure of an Isabelle proof is noteworthy - it is often closer to how a real mathematician would write rather than pure code, making it easier for humans to understand.

Algebraic Geometry

Oddly absent from the Isabelle archive is one of the most popular areas of research in modern mathematics - Algebraic Geometry. The subject evolved from the old observation that equations can be used to describe curves and surfaces into a powerful abstract machinery capable of relating geometric data to algebraic data (in the form of structures such as groups, rings and modules). Central to the modern approach is the concept of *sheaf*, which we now formalised in Isabelle as a certain locale, along with constructions such that of the stalk of a (pre)sheaf.
It is hoped that once a complete formalisation has been attained, researchers will be able to resort to Isabelle's assistance in checking proofs, (counter)examples and conjectures. Besides applications to Physics, Algebraic Geometry has strong ties with Number Theory, which in turn is relevant to Cryptography. The present formalisation follows the Cambridge Part III course on the subject.

The Project

The goal of this project is to expand the existing library of basic mathematical theories and results for the Isabelle proof assistant - doing so enables future researchers to start from there and thus employ tools from the database whenever necessary.
Although a lot has already been added some essential areas are still missing and it is vital for such gaps to be filled in order for interactive theorem provers to become attractive to the general mathematical community as a computational tool. It is an ongoing effort of Prof. Lawrence Paulson at the Computer Laboratory to address this issue and the PMP Projects proved to be a remarkably good way to do so, by capitalizing on the knowledge of mathematicians themselves allied to the technical know-how of computer scientists - this had led to formalisations of results in Linear Algebra and now we move towards the basics of Algebraic Geometry. Such an Isabelle formalisation lends researchers the computational power to better address some kinds of conjectures, as well as look for counterexamples, verify and sometimes simplify proofs.
This iteration of the project made use of an Isabelle feature called a *locale declaration*, which was especially developed to handle axiomatic theories of the sort encountered in abstract algebra. Locales enable one to set parameters and properties that such parameters have to satisfy, creating a *context* within which the computer can identify constructions and facts - this was particularly well-suited for the formalisation of sheaves and morphisms of sheaves in Algebraic Geometry and related developments.

```
locale presheaf = topology +
fixes
opcatopensets ::" ('a) PosetalCategory" and
objectsmap :: "'a set ⇒ ('a, 'm) Ring_scheme" and
restrictionsmap:: "('a set ×'a set) ⇒ ('a  ⇒ 'a)"
assumes
"opcatopensets  ≡ ⦇ Obj = {x. x ∈ T} , Mor = {(x,y)| x y. (x,y) ∈ revpo} ,
 Dom = psheafdom , Cod  = psheafcod , Id  = psheafid  , Comp = psheafcomp ⦈" and
"⋀y w. w ≠ y ⟶ (psheafcomp (x,y) (w,z) = undefined)" and
"⋀x. x ∉ T ⟶ (objectsmap x = undefined)" and
homim: "∀x y.(restrictionsmap (x,y)) ∈ rHom (objectsmap x) (objectsmap y)" and
"⋀ x y . (restrictionsmap (x,x) y = y) " and
functoriality: "∀ x y z .  ( (restrictionsmap (y,z))∘(restrictionsmap (x,y)) =
 restrictionsmap(x,z) )" and
objectsmaptorings: "⋀U. Ring (objectsmap U)"
```

Figure 2: Locale *presheaf* in the formalisation of Algebraic Geometry

About the Author

José Siqueira is a Brazilian mathematician mainly concerned with pure and applied Category Theory and the Foundations of Mathematics, with a side interest in certain flavours of Mathematical Physics and Geometry. The project was conducted from the University of Cambridge Computer Laboratory under the supervision of Prof. Lawrence Paulson, with the support of both the Computer Laboratory and the Faculty of Mathematics. Special thanks are due to Dr. Marjorie Batchelor and Dr. Jacob Rasmussen for making this possible.