Define a group G as

$$G = \{g : g = (g_1, g_2, g_3, g_4), g_\mu \in \{0, 1\}\} \tag{1}$$

with the group operation being the componentwise addition modulo 2, that is

$$g^i g^j = g^j g^i = (g^i_1 + g^j_1, g^i_2 + g^j_2, g^i_3 + g^j_3, g^i_4 + g^j_4) \,(\mathrm{mod}\, 2) \tag{2}$$

For each $g \in G$, define a transformation $d(g)$ on the quark field and the antiquark field as

$$\begin{aligned} d(g)(\psi(x)) &= e^{ix.\pi_g} M_g \psi(x) \\ d(g)(\bar{\psi}(x)) &= e^{ix.\pi_g} \bar{\psi}(x) M_g^\dagger \end{aligned} \tag{3}$$

and its negative counterpart $-d(g)$ as

$$\begin{aligned} -d(g)(\psi(x)) &= -e^{ix.\pi_g} M_g \psi(x) \\ -d(g)(\bar{\psi}(x)) &= -e^{ix.\pi_g} \bar{\psi}(x) M_g^\dagger \end{aligned} \tag{4}$$

whereby π_g are the 16 corners of the Brillouin zone

$$\pi_g = \frac{\pi}{a} g \tag{5}$$

and M_g are the matrices defined as

$$M_g = \prod_{\mu : g_\mu = 1} M_\mu \tag{6}$$

with

$$M_\mu = i\gamma_5\gamma_\mu \tag{7}$$

The naive action for free fermions on the lattice given by

$$S_0(\psi) = a^4 \sum_x \{\sum_\mu \bar{\psi}(x)\gamma_\mu \frac{1}{2a}[\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})] + m\bar{\psi}(x)\psi(x)\} \tag{8}$$

is invariant under this set of 32 discrete transformations. We note that these transformations compose with one another according to the following

$$\begin{aligned} d(g^i) \circ d(g^j)(\psi(x)) &= e^{ix.(\pi_{g^i} + \pi_{g^j})} M_{g^i} M_{g^j} \psi(x) = \varsigma_{ij} e^{ix.\pi_{g^i g^j}} M_{g^i g^j} \psi(x) \\ d(g^i) \circ d(g^j)(\bar{\psi}(x)) &= e^{ix.(\pi_{g^i} + \pi_{g^j})} \bar{\psi}(x) M^\dagger_{g^j} M^\dagger_{g^i} = \varsigma_{ij} e^{ix.\pi_{g^i g^j}} \bar{\psi}(x) M^\dagger_{g^i g^j} \end{aligned} \tag{9}$$

where $\varsigma_{ij} \in \{\pm 1\}$ are such that

$$M_{g^i} M_{g^j} = \varsigma_{ij} M_{g^i g^j} \tag{10}$$

We see that the 32 transformations given in (3) and (4) form a group, the "doubling symmetry" group D, with its structure inherited from the group G such that

$$D = \{\pm d(g) : d(g^i)d(g^j) = \varsigma_{ij} d(g^i g^j), g \in G\} \tag{11}$$

In other words, we have

$$q : D \to D/\{\pm I_D\} \cong G \tag{12}$$

We are interested in finding irreducible representations of the doubling symmetry group D. To proceed, we would first like to look at the irreps of group G, which can then be lifted up to irreps of D by composing with the quotient map q from (12). To determine the irreps of G, we make use of its following properties

(1) G is an abelian group of order 16

(2) all group elements of G, except the identity, have order 2

(3) G is generated by its 4 elements $g^1 = (1, 0, 0, 0)$, $g^2 = (0, 1, 0, 0)$, $g^3 = (0, 0, 1, 0)$, $g^4 = (0, 0, 0, 1)$

By property (1), G has 16 inequivalent 1-dimensional irreps. By property (2) such an irrep can only go to itself or be multiplied by a minus sign under any group element of G. By property (3) each irrep of G is uniquely determined by how it transforms under g^1, g^2, g^3, g^4. Therefore we label the 16 irreps of G, $\rho^1_G(\xi)$, by a 4-component vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4), \qquad \xi_\mu \in \{\pm 1\} \tag{13}$$

such that the corresponding vector space $\langle v(\xi)\rangle$ transforms under G according to

$$\rho^1_G(\xi)(g^\mu) : v(\xi) \mapsto \xi_\mu v(\xi), \qquad \mu \in \{1, 2, 3, 4\} \tag{14}$$

Lifting up, we get 16 1-dimensional irreps of D, $\rho_D^1(\xi)$ on the vector space $\langle v(\xi)\rangle$, such that

$$\rho_D^1(\xi)(\pm d(g)) : v(\xi) \mapsto v(\xi) \prod_{\mu : g_\mu = 1} \xi_\mu \tag{15}$$

Define a matrix group M as

$$M = \{\pm M_g : g \in G\} \tag{16}$$

with the group operation being the usual matrix multiplication, we have

$$D \cong M \tag{17}$$

The doubling symmetry group D breaks into 17 conjugacy classes

$$\{\pm d(g)\}_{g \in G \setminus \{I_G\}} \bigcup \{-I_D\} \bigcup \{I_D\} \tag{18}$$

and as for a finite group number of irreps equals number of conjugacy classes, we deduce that there is a last 4-dimensional irrep of D, denoted by ρ_D^4 such that

$$\rho_D^4(\pm d(g)) = \pm M_g \tag{19}$$

and obtain the full character table of the doubling symmetry group D as follows

	$\leftarrow \{\pm d(g)\}_{g \in G \setminus \{I_G\}} \rightarrow$	$\{-I_D\}$	$\{I_D\}$
$\uparrow$ $\rho_D^1(\xi)$ $\downarrow$	$\prod_{\mu : g_\mu = 1} \xi_\mu$	1	1
ρ_D^4	$\leftarrow 0 \rightarrow$	-4	4

where the first row lists the 17 conjugacy classes of D, and the first column lists its 17 irreps.

We return to the representation of D on the quark field. This representation has no overlap with any of the irreps $\rho_D^1(\xi)$, because the identification of $-I_D$ to I_D in the 1-dimensional irreps is unphysical. Therefore the representation of D on the quark field reduces to copies of irrep ρ_D^4 , and by the same reasoning so does the representation of D on the antiquark field.

To analyse the diquark representation and the antiquark-quark representation, it suffices to look at $\rho_D^4 \otimes \rho_D^4$. Recalling that the character of a tensor product representation is the product of the characters of its factors, we compute the character of $\rho_D^4 \otimes \rho_D^4$ as

	$\leftarrow \{\pm d(g)\}_{g \in G \setminus \{I_G\}} \rightarrow$	$\{-I_D\}$	$\{I_D\}$
$\rho_D^4 \otimes \rho_D^4$	$\leftarrow 0 \rightarrow$	16	16

Employing the projection formula for the multiplicity of irrep $\rho_D^1(\xi)$ in the representation $\rho_D^4 \otimes \rho_D^4$

$$m_{\rho_D^1(\xi)}^{\rho_D^4 \otimes \rho_D^4} = \langle \chi_{\rho_D^1(\xi)}, \chi_{\rho_D^4 \otimes \rho_D^4} \rangle \tag{20}$$

whereby the inner product $\langle\ ,\ \rangle$ on the characters of any two representations, α and β, for a general finite group Ω is defined as

$$\langle \chi_\alpha, \chi_\beta \rangle = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \overline{\chi_\alpha(\omega)} \chi_\beta(\omega) \tag{21}$$

we decompose the tensor product representation into

$$\rho_D^4 \otimes \rho_D^4 = \oplus_\xi \rho_D^1(\xi) \tag{22}$$

i.e. the diquark representation and the antiquark-quark representation are both described by the 16 1-dimensional irreps of the doubling symmetry group.

We look at meson operators of the form

$$\bar{\psi}(x)\Gamma^m\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r), \qquad 0\le r\le 4 \tag{23}$$

whereby $\hat{\mu}_1,\cdots,\hat{\mu}_r$ are distinct links, i.e. space-time vectors of unit length along directions given by the indices $\mu_1,\cdots,\mu_r$ respectively. Making use of the identity

$$M_g^\dagger\gamma_{u_1}\cdots\gamma_{u_k}M_g=(-1)^{g_{u_1}+\cdots+g_{\mu_k}}\gamma_{u_1}\cdots\gamma_{u_k}, \qquad \mu_1,\cdots,\mu_k\in\{1,2,3,4\} \tag{24}$$

we establish, for each choice of links, a one-to-one correspondence between the gamma matrices $\Gamma^m_{\mu_1\cdots\mu_r}$ and the irreps $\rho^1_D(\xi)$ of the antiquark-quark representation

$$\Gamma^m_{\mu_1\cdots\mu_r}(\xi)=\gamma_{u_1}\cdots\gamma_{u_r}\prod_{\mu:\xi_\mu=-1}\gamma_u \quad\Leftrightarrow\quad \rho^1_D(\xi) \tag{25}$$

Explcitly, we have the above relation as

$$\begin{aligned}\pm d(g):\bar{\psi}(x)\Gamma^m_{\mu_1\cdots\mu_r}(\xi)\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r) &\mapsto (-1)^{g_{u_1}+\cdots+g_{\mu_r}}\bar{\psi}(x)M_g^\dagger\Gamma^m_{\mu_1\cdots\mu_r}(\xi)M_g\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r)\\ &= \bar{\psi}(x)\Gamma^m_{\mu_1\cdots\mu_r}(\xi)\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r)\prod_{\mu:g_\mu=1}\xi_\mu\end{aligned} \tag{26}$$

which is just the definition for irrep $\rho^1_D(\xi)$ given in (15).

For diquark operators of the form

$$\psi^T(x)\Gamma^b\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r), \qquad 0\le r\le 4 \tag{27}$$

we make use of the identity modified from (24)

$$M_g^T\gamma_{u_1}\cdots\gamma_{u_k}M_g=(-1)^{g_2+g_4}(-1)^{g_{u_1}+\cdots+g_{\mu_k}}\gamma_{u_1}\cdots\gamma_{u_k}, \qquad \mu_1,\cdots,\mu_k\in\{1,2,3,4\} \tag{28}$$

and obtain a similar one-to-one correspondence, for each choice of links, between the gamma matrices $\Gamma^b_{\mu_1\cdots\mu_r}$ and the irreps $\rho^1_D(\xi)$ of the diquark representation

$$\Gamma^b_{\mu_1\cdots\mu_r}(\xi)=\gamma_2\gamma_4\gamma_{u_1}\cdots\gamma_{u_r}\prod_{\mu:\xi_\mu=-1}\gamma_u \quad\Leftrightarrow\quad \rho^1_D(\xi) \tag{29}$$

or explicitly

$$\begin{aligned}\pm d(g):\psi^T(x)\Gamma^b_{\mu_1\cdots\mu_r}(\xi)\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r) &\mapsto (-1)^{g_{u_1}+\cdots+g_{\mu_r}}\psi^T(x)M_g^T\Gamma^b_{\mu_1\cdots\mu_r}(\xi)M_g\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r)\\ &= \psi^T(x)\Gamma^b_{\mu_1\cdots\mu_r}(\xi)\psi(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_r)\prod_{\mu:g_\mu=1}\xi_\mu\end{aligned} \tag{30}$$

Another symmetry of the free fermion action (8) is the translational symmetry generated by single lattice spacing shifts $t(\hat{\nu})$ in the spatial directions given by $\nu\in\{1,2,3\}$

$$\begin{aligned}t(\hat{\nu})(\psi(x)) &= \psi(x+a\hat{\nu})\\ t(\hat{\nu})(\bar{\psi}(x)) &= \bar{\psi}(x+a\hat{\nu})\end{aligned} \tag{31}$$

and to be consistent with the periodic boundary conditions we must have $t(\hat{\nu})^N$ go to identity for $\nu\in\{1,2,3\}$. Therefore these shifts form a translational symmetry group T

$$T=\{t(n):t(n)=\prod_\nu t(\hat{\nu})^{n_\nu},n_\nu\in\mathbb{Z}_N\} \tag{32}$$

which is an abelian group with its group operation defined as

$$t(n^i)t(n^j)=t(n^j)t(n^i)=t(n^i+n^j), \qquad (n^i+n^j)_\nu=n^i_\nu+n^j_\nu\,(mod\,N) \tag{33}$$

and isomorphic to the direct product of cyclic group $\mathbb{Z}_N$

$$T\cong(\mathbb{Z}_N)^3 \tag{34}$$

T has N^3 1-dimensional irreps $\rho^1_T(p)$, labelled by a 3-component vector p, the momentum

$$p=(p_1,p_2,p_3), \qquad p_\nu\in\frac{2\pi}{N}\{0,\cdots,N-1\} \tag{35}$$

such that the corresponding vector space $\langle w(p)\rangle$ transforms under T according to

$$\rho^1_T(p)(t(\hat{\nu})) : w(p) \mapsto e^{ip_\nu} w(p) \tag{36}$$

Define a larger lattice symmetry group S which incorporates both the doubling symmetry and the translational symmetry

$$S = \langle d(g^\mu), t(\hat{\nu})\rangle^{\mu\in\{1,2,3,4\}}_{\nu\in\{1,2,3\}} \tag{37}$$

we note its group elements $\{\pm d(g)\}_{g\in G}$ either commute or anticommute with $\{t(n)\}_{n_\nu\in\mathbb{Z}_N}$ according to

$$t(n)d(g)t(n)^{-1} = (-1)^{\sum_\nu n_\nu g_\nu} d(g) \tag{38}$$

and therefore deduce the following properties of S

(1) $Z(S) = \{\pm t(n)\}_{n_\nu\in 2\mathbb{Z}_N \forall\nu} \bigcup \{\pm d(g^4)t(n)\}_{n_\nu\notin 2\mathbb{Z}_N\forall\nu}$
(2) $S/\{\pm I_S\} \cong G\times T$
(3) $S = D \rtimes T$

By property (1) we know S breaks into $(16+\frac{1}{4})N^3$ conjugacy classes

$$\{\pm d(g)t(n)\}_{d(g)t(n)\notin Z(S)} \bigcup \{-d(g)t(n)\}_{d(g)t(n)\in Z(S)} \bigcup \{d(g)t(n)\}_{d(g)t(n)\in Z(S)} \tag{39}$$

By property (2) we get $16N^3$ 1-dimensional irreps of S lifted up from the 1-dimensional irreps of the abelian group $G\times T$ in a similar fashion as in our treatment of group D. Explicitly, the $16N^3$ irreps of $G\times T$ are just the tensor products of irreps of G and irreps of T, labelled by a pair of vectors (ξ, p)

$$\rho^1_{G\times T}(\xi,p) = \rho^1_G(\xi)\otimes\rho^1_T(p) \qquad \Leftrightarrow \qquad \rho^1_{G\times T}(\xi,p)(g,t(n)) : v(\xi)\otimes w(p) \mapsto v(\xi)\otimes w(p) \prod_{\mu:g_\mu=1} \xi_\mu e^{i\sum_\nu n_\nu p_\nu} \tag{40}$$

and the corresponding 1-dimensional irreps of S, $\rho^1_S(\xi,p)$, have characters

	$\{\pm d(g)t(n)\}_{d(g)t(n)\notin Z(S)}$	$\{-d(g)t(n)\}_{d(g)t(n)\in Z(S)}$	$\{d(g)t(n)\}_{d(g)t(n)\in Z(S)}$
$\uparrow$ $\rho^1_S(\xi,p)$ $\downarrow$	$\prod_{\mu:g_\mu=1}\xi_\mu e^{i\sum_\nu n_\nu p_\nu}$	$\prod_{\mu:g_\mu=1}\xi_\mu e^{i\sum_\nu n_\nu p_\nu}$	$\prod_{\mu:g_\mu=1}\xi_\mu e^{i\sum_\nu n_\nu p_\nu}$

To find the remaining $\frac{1}{4}N^3$ irreps of S, we make use of relation (38), which gives a homomorphism θ from T to the group of automorphisms of D

$$\theta(t(n)) : \pm d(g) \mapsto \pm t(n)d(g)t(n)^{-1} \tag{41}$$

Since D has a single 4-dimensional irrep, ρ^4_D, given $t(n)$ we have $\rho^4_D\circ\theta(t(n))$ as a 4-dimensional irrep of D equivalent to ρ^4_D, and some 4×4 transformation matrix $P(t(n))$ such that

$$P(t(n))\rho^4_D(\pm d(g))P(t(n))^{-1} = \rho^4_D\circ\theta(t(n))(\pm d(g)) \qquad \Leftrightarrow \qquad P(t(n))M_\mu P(t(n))^{-1} = (-1)^{n_\mu} M_\mu \tag{42}$$

whereby n_4 is taken to be identically 0. By Schur's lemma, the transformation matrices $P(t(n))$ form a projective representation of T, and in particular the choice ϱ^4_T defined as

$$\begin{aligned} \varrho^4_T(t(\hat{\nu})) &= \gamma_\nu \\ \varrho^4_T(t(n)) &= \prod_\nu \gamma_\nu^{n_\nu} = \prod_{\nu:n_\nu\notin 2\mathbb{Z}_N} \gamma_\nu \end{aligned} \tag{43}$$

has its factor set f_T taking values in $\{\pm 1\}$

$$\varrho^4_T(t(n^i))\varrho^4_T(t(n^j)) = f_T(t(n^i),t(n^j))\varrho^4_T(t(n^i)t(n^j)), \qquad f_T(t(n^i),t(n^j))\in\{\pm1\} \quad \forall\,(n^i,n^j) \tag{44}$$

By property (3) S has the structure

$$S = \{\pm d(g)t(n) : d(g^i)t(n^i)d(g^j)t(n^j) = d(g^i)\theta(t(n^i))(d(g^j))t(n^i)t(n^j)\} \tag{45}$$

therefore by (42) the map ϱ^4_S defined as

$$\varrho^4_S(\pm d(g)t(n)) = \rho^4_D(\pm d(g))\varrho^4_T(t(n)) \tag{46}$$

is a projective representation of S, with its factor set f_S inherited from that of ϱ_T^4

$$f_S(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j)) = f_T(t(n^i), t(n^j)) \tag{47}$$

Since the 4-dimensional projective representation of T, ϱ_T^4, given by (43) maps the generators of the abelian group, $\{t(\hat{\nu})\}_{\nu\in\{1,2,3\}}$, to anticommuting matrices, we are inspired to find a 2-dimensional projective representation of T, ϱ_T^2, defined as

$$\begin{aligned} \varrho_T^2(t(\hat{\nu})) &= \sigma_\nu \\ \varrho_T^2(t(n)) &= \prod_\nu \sigma_\nu^{n_\nu} = \prod_{\nu: n_\nu \notin 2\mathbb{Z}_N} \sigma_\nu \end{aligned} \tag{48}$$

such that it has the same factor set as that of ϱ_T^4, f_T. By property (3) we also have

$$q : S \to S/D \cong T \tag{49}$$

therefore we can lift ϱ_T^2 up to the corresponding 2-dimensional projective representation of S, ϱ_S^2, with the same factor set as that of ϱ_S^4, f_S. Similarly we can lift $\rho_T^1(p)$ up to $\rho_S^1(p)$ as a 1-dimensional true representation of S. Finally, we define an 8-dimensional projective representation of S, $\rho_S^8(p)$, as the tensor product

$$\rho_S^8(p) = \rho_S^1(p) \otimes \varrho_S^2 \otimes \varrho_S^4 \tag{50}$$

and see that it has the trivial factor set f_S^*

$$f_S^*(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j)) = f_S(\pm d(g^i)t(n^i), \pm d(g^j)t(n^j))^2 \equiv 1 \tag{51}$$

and is thus in fact a true representation of S. Explicitly, we have the group elements of S represented as the 8×8 matrices

$$\rho_S^8(p)(\pm d(g)t(n)) = \pm e^{i\sum_\nu n_\nu p_\nu} \prod_{\nu: n_\nu \notin 2\mathbb{Z}_N} \sigma_\nu \otimes M_g \prod_{\nu: n_\nu \notin 2\mathbb{Z}_N} \gamma_\nu \tag{52}$$

and taking the product of traces of the matrices $\varrho_S^2(\pm d(g)t(n))$ and $\varrho_S^4(\pm d(g)t(n))$, we obtain the character of $\rho_S^8(p)$ as

	$\{\pm d(g)t(n)\}_{d(g)t(n)\notin Z(S)}$	$\{-d(g)t(n)\}_{d(g)t(n)\in Z(S)}$	$\{d(g)t(n)\}_{d(g)t(n)\in Z(S)}$
$\rho_S^8(p)$	0	$-8e^{i\sum_\nu n_\nu p_\nu}$	$8e^{i\sum_\nu n_\nu p_\nu}$

$\rho_S^8(p)$ is irreducible as its character obeys the relation

$$\langle \chi_{\rho_S^8(p)}, \chi_{\rho_S^8(p)} \rangle = 1 \tag{53}$$

For each p in the range given by $p_\nu \in \frac{2\pi}{N}\{0, \cdots, \frac{N}{2}-1\}$, define a new 8-dimensional irrep of S from $\rho_S^8(p)$, denoted by $\tilde{\rho}_S^8(p)$, as

$$\tilde{\rho}_S^8(p) = \rho_S^1(\breve{\pi}) \otimes \rho_S^8(p) \tag{54}$$

whereby $\breve{\pi}$ denotes the 3-component vector (π, π, π). These $\frac{1}{4}N^3$ irreps

$$\{\rho_S^8(p)\} \bigcup \{\tilde{\rho}_S^8(p)\}, \qquad p_\nu \in \frac{2\pi}{N}\{0, \cdots, \frac{N}{2}-1\} \tag{55}$$

give distinct characters and are thus pairwise inequivalent. We therefore exhaust the remaining irreps of S, and obtain the full character table of group S as follows

	$\{\pm d(g)t(n)\}_{d(g)t(n)\notin Z(S)}$	$\{-d(g)t(n)\}_{d(g)t(n)\in Z(S)}$	$\{d(g)t(n)\}_{d(g)t(n)\in Z(S)}$
$\uparrow$ $\rho_S^1(\xi, p)$ $\downarrow$	$\prod_{\mu:g_\mu=1} \xi_\mu e^{i\sum_\nu n_\nu p_\nu}$	$\prod_{\mu:g_\mu=1} \xi_\mu e^{i\sum_\nu n_\nu p_\nu}$	$\prod_{\mu:g_\mu=1} \xi_\mu e^{i\sum_\nu n_\nu p_\nu}$
$\uparrow$ $\rho_S^8(\grave{p})$ $\downarrow$	0	$-8e^{i\sum_\nu n_\nu \grave{p}_\nu}$	$8e^{i\sum_\nu n_\nu \grave{p}_\nu}$
$\uparrow$ $\tilde{\rho}_S^8(\grave{p})$ $\downarrow$	0	$-8(-1)^{\sum_\nu n_\nu} e^{i\sum_\nu n_\nu \grave{p}_\nu}$	$8(-1)^{\sum_\nu n_\nu} e^{i\sum_\nu n_\nu \grave{p}_\nu}$

with p in the range given by $p_\nu \in \frac{2\pi}{N}\{0,\cdots,N-1\}$ and $\grave{p}$ in the range given by $\grave{p}_\nu \in \frac{2\pi}{N}\{0,\cdots,\frac{N}{2}-1\}$. For each $\grave{p}$, both $\rho^8_S(\grave{p})$ and $\tilde{\rho}^8_S(\grave{p})$ give an 8-dimensional representation of T as a subgroup of S, denoted by $\rho^8_S(\grave{p})|_T$ and $\tilde{\rho}^8_S(\grave{p})|_T$ respectively. Modifying (52) we have

$$\begin{aligned}
\rho^8_S(\grave{p})|_T(t(n)) &= e^{i\sum_\nu n_\nu \grave{p}_\nu} \prod_{\nu:n_\nu\notin 2\mathbb{Z}_N} \sigma_\nu \otimes \prod_{\nu:n_\nu\notin 2\mathbb{Z}_N} \gamma_\nu \\
\tilde{\rho}^8_S(\grave{p})|_T(t(n)) &= (-1)^{\sum_\nu n_\nu} e^{i\sum_\nu n_\nu \grave{p}_\nu} \prod_{\nu:n_\nu\notin 2\mathbb{Z}_N} \sigma_\nu \otimes \prod_{\nu:n_\nu\notin 2\mathbb{Z}_N} \gamma_\nu
\end{aligned} \tag{56}$$

and employing the projection formulas for the multiplicities of irreps of T in $\rho^8_S(\grave{p})|_T$ and $\tilde{\rho}^8_S(\grave{p})|_T$

$$\begin{aligned}
m^{\rho^8_S(\grave{p})|_T}_{\rho^1_T(p)} &= \langle \chi_{\rho^1_T(p)}, \chi_{\rho^8_S(\grave{p})|_T}\rangle = \frac{8}{N^3}\prod_\nu \sum_{n_\nu\in 2\mathbb{Z}_N} e^{in_\nu(\grave{p}_\nu - p_\nu)} \\
m^{\tilde{\rho}^8_S(\grave{p})|_T}_{\rho^1_T(p)} &= \langle \chi_{\rho^1_T(p)}, \chi_{\tilde{\rho}^8_S(\grave{p})|_T}\rangle = \frac{8}{N^3}\prod_\nu \sum_{n_\nu\in 2\mathbb{Z}_N} e^{in_\nu(\grave{p}_\nu - p_\nu)}
\end{aligned} \tag{57}$$

we deduce

$$\rho^8_S(\grave{p})|_T \cong \tilde{\rho}^8_S(\grave{p})|_T = \oplus_{p:p=\grave{p}+\dot{\pi}} \rho^1_T(p) \tag{58}$$

whereby $\dot{\pi}$ denotes a 3-component vector such that $\dot{\pi}_\nu \in \{0,\pi\}$. In other words, given $\grave{p}$ with $\grave{p}_\nu \in \frac{2\pi}{N}\{0,\cdots,\frac{N}{2}-1\}$, the two 8-dimensional irreps of S associated with $\grave{p}$, $\rho^8_S(\grave{p})$ and $\tilde{\rho}^8_S(\grave{p})$, both couple to the eight 1-dimensional irreps of T, $\{\rho^1_T(\grave{p}+\dot{\pi})\}_{\dot{\pi}_\nu\in\{0,\pi\}}$, but are nonetheless inequivalent representations of the whole group S taking into account the doubling symmetry.

On the quark field, the identification of $-I_S$ to I_S in the 1-dimensional irreps of S, $\rho^1_S(\xi,p)$, is unphysical, thereby we conclude that the representation of S on the quark field decomposes into copies of $\rho^8_S(\grave{p})$ and $\tilde{\rho}^8_S(\grave{p})$ for different $\grave{p}$ in the range given by $\grave{p}_\nu \in \frac{2\pi}{N}\{0,\cdots,\frac{N}{2}-1\}$. By the same reasoning the representation of S on the antiquark field decomposes into copies of the 8-dimensional irreps of S alike.

To analyse the diquark representation and the antiquark-quark representation, it suffices to look at the four cases, $\rho^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j)$, $\rho^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j)$, $\tilde{\rho}^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j)$, and $\tilde{\rho}^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j)$, for a particular choice of $\grave{p}^i$ and $\grave{p}^j$. Taking the products of characters of the factors we compute the characters of these four representations as follows

	$\{\pm d(g)t(n)\}_{d(g)t(n)\notin Z(S)}$	$\{-d(g)t(n)\}_{d(g)t(n)\in Z(S)}$	$\{d(g)t(n)\}_{d(g)t(n)\in Z(S)}$
$\rho^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j)\cong\tilde{\rho}^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j)$	0	$64e^{i\sum_\nu n_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu)}$	$64e^{i\sum_\nu n_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu)}$
$\rho^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j)\cong\tilde{\rho}^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j)$	0	$64(-1)^{\sum_\nu n_\nu}e^{i\sum_\nu n_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu)}$	$64(-1)^{\sum_\nu n_\nu}e^{i\sum_\nu n_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu)}$

Denoting

$$\begin{aligned}
\rho^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j)\cong\tilde{\rho}^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j) &= \zeta(\grave{p}^i,\grave{p}^j) \\
\rho^8_S(\grave{p}^i)\otimes\tilde{\rho}^8_S(\grave{p}^j)\cong\tilde{\rho}^8_S(\grave{p}^i)\otimes\rho^8_S(\grave{p}^j) &= \zeta'(\grave{p}^i,\grave{p}^j)
\end{aligned} \tag{59}$$

we have the overlaps between $\zeta(\grave{p}^i,\grave{p}^j)$, $\zeta'(\grave{p}^i,\grave{p}^j)$ and the 1-dimensional irreps of S, $\rho^1_S(\xi,p)$, to be

$$\begin{aligned}
m^{\zeta(\grave{p}^i,\grave{p}^j)}_{\rho^1_S(\xi,p)} &= \langle \chi_{\rho^1_S(\xi,p)}, \chi_{\zeta(\grave{p}^i,\grave{p}^j)}\rangle = \frac{4}{N^3}(1+\xi_4 e^{i\sum_\nu \grave{p}^i_\nu+\grave{p}^j_\nu-p_\nu})\prod_\nu\sum_{n_\nu\in 2\mathbb{Z}_N} e^{in_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu-p_\nu)} \\
m^{\zeta'(\grave{p}^i,\grave{p}^j)}_{\rho^1_S(\xi,p)} &= \langle \chi_{\rho^1_S(\xi,p)}, \chi_{\zeta'(\grave{p}^i,\grave{p}^j)}\rangle = \frac{4}{N^3}(1-\xi_4 e^{i\sum_\nu \grave{p}^i_\nu+\grave{p}^j_\nu-p_\nu})\prod_\nu\sum_{n_\nu\in 2\mathbb{Z}_N} e^{in_\nu(\grave{p}^i_\nu+\grave{p}^j_\nu-p_\nu)}
\end{aligned} \tag{60}$$

therefore $\zeta(\grave{p}^i,\grave{p}^j)$ and $\zeta'(\grave{p}^i,\grave{p}^j)$ each reduce to 64 1-dimensional irreps of S associated with $\grave{p}^i+\grave{p}^j$ according to

$$\begin{aligned}
\zeta(\grave{p}^i,\grave{p}^j) &= \oplus_{p:p=\grave{p}^i+\grave{p}^j+\dot{\pi}}(\oplus_{\xi:\xi_4=e^{i\sum_\nu\dot{\pi}_\nu}}\rho^1_S(\xi,p)) \\
\zeta'(\grave{p}^i,\grave{p}^j) &= \oplus_{p:p=\grave{p}^i+\grave{p}^j+\dot{\pi}}(\oplus_{\xi:\xi_4=-e^{i\sum_\nu\dot{\pi}_\nu}}\rho^1_S(\xi,p))
\end{aligned} \tag{61}$$

while the two sets of irreps from $\{\rho^1_S(\xi,p)\}_{p=\grave{p}^i+\grave{p}^j+\dot{\pi}}$ do not overlap with each other.