George Peacock and the Development of British Algebra 1800-1840

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Introduction

The story of British algebra in the first part of the nineteenth century is one where there is little influence from the continent. Although British writers are reading works that are published on the continent, there does not seem to be much on the foundations of algebra that is read by the British mathematicians. Between about 1800 and 1820 this was largely to do with the insularity that British mathematics had developed over the course of the previous century. After 1820, some works were being translated and read, but writers on the continent were not concerned with the same issues that the British algebraists were debating.

Up until 1830 there are two fairly distinct stories: the first concerning the development of algebra as a rigourous science, and the second the ways in which the complex numbers could be geometrised. The former of these stories is concerned firstly with the writings of Woodhouse, not the more famous contribution of his where he was the first person to publish in Britain using Leibnitzian notation for differentials, but his considerations of rigourising complex numbers. This story then moves on to consider the work done by the Analytical Society, in particular an unpublished paper of Charles Babbage which marks the first real attempt to put a general algebra onto a secure footing. The geometry story is more natural, with the development of the Argand diagram in Britain: it had already been developed on the continent.

These two threads are linked together by the writings of George Peacock, in particular his *Treatise on Algebra*[1], which is the first work to consider both together. The major concept of Peacock's that is well known is his principle of the permanence of equivalent forms, which is a part of the system that he develops in order to attempt to place algebra on a rigourous footing. Peacock expanded upon the ideas in his *Report on the Recent Developments and Present State of Certain Branches of Analysis*[2], which, read in 1833 and then published in 1834. He gave a further clarification in the second edition of his *Treatise*[3], published in two volumes in 1842 and 1845. By this point, however, Peacock was less interested in mathematics than he had been before, as he had become Dean of Ely Cathedral.[4]

After Peacock, there were a number of criticisms published of his work, which seek to establish other ways of doing algebra, establishing some reasonable faults in his work. However, the next major work in algebra, that of De Morgan, was not so much a criticism of Peacock's work, but an attempt to build upon it, but this is beyond the scope of this essay.

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[1] (Peacock, 1830)
[2] (Peacock, 1834)
[3] (Peacock, 1842) and (Peacock, 1845)
[4] (Venn & Venn, 1940-54)

Algebra before Peacock

Algebra in 1800

The story of algebra at the end of the nineteenth century in Britain was one in which the significant mathematicians of the day were against any use of negative or complex numbers. The main proponents of this theory were mathematicians such as Robert Simson, Baron Francis Maseres and William Frend[5]. The uneasiness that these authors felt was that there was no justification for using these numbers, and that conclusions reached through their use would not be acceptable. In his 1796 work, *Principles of Algebra*, he argued that the previous explanations of negative and complex numbers are not able to justify their principles *'without reference to metaphor, the probability is, that he has never thought accurately on the subject.'*[6]

This opposition to the idea of negative numbers was founded on viewing arithmetic as based on the concept of quantities that can be observed. Lines have lengths, for example, that are simply non-negative real numbers, but it wass not clear to this generation of mathematicians what a negative length was. As a consequence of this, operations such as subtraction and the taking of roots were restricted to ensure that the solutions reached were solely arithmetical. Indeed, Frend said later that *'it submits to be taken away from another number greater than itself, but to attempt to take it away from a number greater than itself is ridiculous'*[7]. The strength of these convictions was a key influence in the subsequent development of algebra; they were an important motivation for the later generations of mathematicians, particularly George Peacock.

This generation also considered algebra to be a form of *universal arithmetic*, where the algebraic equations merely used letters to represent certain numbers that were known or unknown. However, the conditions on these numbers needed to be met, and in cases where the solution to a problem was negative, this meant that the problem was incorrectly stated, and it was required to rewrite it to give the answer as a positive number.

Woodhouse

The first person in Britain to challenge this view was Robert Woodhouse with his 1801 paper *On the Necessary Truth of Certain Conclusions Obtained by Means of Imaginary Quantities*[8]. The principal importance of the paper is not in the methods that Woodhouse introduces, but rather in the way in which this, and more importantly Woodhouse himself, influenced the next generation of mathematicians. Woodhouse set himself up as responding to a work of Playfair from 1778[9], but it is his influence in trying to establish a way of rigourising complex numbers which is important. He

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[5] For example, see (Pycior, 1981, pp. 27-30) or (Fisch, 1994, pp. 262-263).
[6] (Frend, 1796, p. x)
[7] Ibid.
[8] (Woodhouse, 1801). It is worth noting that this is occasionally referenced as being published in 1802, for example (Becher, 1980).
[9] *On the Arithmetic of Impossible Quantities*. See (Sherry, 1991) and (Pycior, 1984) for more on this.

followed this up with a paper in 1803 (*The Principles of Analytical* Calculation)[10] in which he expounds some more of his theory.

Playfair had discussed a method involving the concept of analogy[11], to which Woodhouse was responding. The main thrust of Woodhouse's argument is that the definitions of operations could be extended to incorporate the use of complex numbers. He argued that:

> *'... $\frac{x\sqrt{-1}}{\sqrt{-1}}$ is equal to x, not because it is true that a quantity multiplied and divided by the same number remains the same, but because $\frac{x\sqrt{-1}}{\sqrt{-1}}$ means, that x is to be combined with $\sqrt{-1}$ after the manner that real numbers are in multiplication, and then divided after the manner that real quantities are in division; and therefore, since the two operations are the reverse of each other, $\frac{x \times \sqrt{-1}}{\sqrt{-1}}$ and x must be equivalent expressions.'*[12]

The concept of extending the definitions can clearly be seen here, although it is important here that the idea was purely to extend the definitions so that they could apply for the particular symbol $\sqrt{-1}$, not for anything more general than this. The relevance of this is that it is the direction in which Babbage and Peacock developed their understanding of algebra.

The particular case that interested Woodhouse was how to think of $e^{x\sqrt{-1}}$, to which he assigned the Taylor series result, and this was the only definition that he allowed. Indeed, this was considered the definition to the extent that Woodhouse says that *'$e^{x\sqrt{-1}}$ is an abridged symbol for $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$ &c.'*, emphasising how Woodhouse was making this the definition of the complex index.

Harvey Becher[13] has contested that Woodhouse had the ideas both of Babbage and Peacock, and that these were doing no more than slightly improving on them. The five key ideas that Becher drew on are those of (Dubbey, 1977):

> *'(1) Algebra had previously been considered only as a modification of arithmetic. (2) Algebra consists of the manipulation of symbols in a way independent of any particular interpretation. (3) Arithmetic is only a special case of Algebra--a "Science of Suggestion" as Peacock put it. (4) The sign "=" is to be taken as meaning "is algebraically equivalent to." (5) The principle of the permanence of equivalent forms.'*[14]

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[10] (Woodhouse, 1803)
[11] See (Pycior, 1984, p. 430).
[12] (Woodhouse, 1801, p. 99)
[13] (Becher, 1980). This is in part a response to (Dubbey, 1977), and both contest that they have discovered a new point at which the principle ideas have been discovered, reducing them to the five ideas given in the text above. However, such a view seems in each case to be optimistic, in that it is more natural to see the ideas being developed by the subsequent authors, rather than any one of them being much more important than the others.
[14] (Becher, 1980, p. 390), quoting (Dubbey, Babbage, Peacock and Modern Algebra, 1977, p. 298).

There is indeed evidence that Woodhouse was thinking of ideas that are akin to those in Peacock, but it is in the subtleties that there is a distinction. Particularly important is considering what the extension of the algebra applies to, which Becher does not mention.

The Analytical Society

A crucial part in this story is played by the Analytical Society in Cambridge. The key players of the Analytical society in the development of algebra are George Peacock and Charles Babbage, and the algebraic story is not developed by anyone outside this group until the middle of the 1830s. This group were studying at Cambridge from about 1811, and many of these became fellows after graduating.[15]

Woodhouse himself was at this time a fellow of Caius College, which is where Edward Bromhead, another key member of the Analytical Society, had been a student.[16] With this influence from Woodhouse[17], there was almost certainly discussion of the algebra he proposed. For example, Bromhead writes to Babbage in 1821, discussing algebra, particularly thoughts he has concerning its development[18].

Charles Babbage and '*The Philosophy of Analysis*'

Not much has been written about Charles Babbage's manuscript entitled *The Philosophy of Analysis*[19] - it is in the British Library's manuscript collection and seems not to be well studied. The major work on this is by Dubbey, and most people who have written about this since then have cited him as their source, rather than the manuscript itself[20]. Dubbey viewed the manuscript solely through the eyes of a scholar of Babbage, in a way akin to Becher on Woodhouse; this is discussed more below in the section on Peacock.

There are indications that Babbage intended to publish it when he wrote it, indeed, at one stage he has an asterisk, referencing a comment that says, *'Inclusion in a note a proof'*[21]. It is not known why he did not publish the work; it was shortly afterwards that he began to build his difference engine[22], so it may just be that he moved on to other things. Indeed, only four of his works were published after 1823, a pair of papers in the Transactions of the Cambridge Philosophical Society in 1827 and another pair of papers in the Edinburgh Encyclopaedia in 1830. However, it is

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[15] There are many publications on the Analytical Society. For example, see (Dubbey, 1978).
[16] (Venn & Venn, 1940-54)
[17] (Becher, 1980, p. 393) gives a convincing argument that Peacock and other members of the Analytical Society were familiar with aspects of Woodhouse's work.
[18] (Bromhead, 1821), as quoted in (Dubbey, 1978, p. 93).
[19] (Babbage, c.1821). This work is cited by Dubbey (in (Dubbey, 1977) and (Dubbey, 1978)) as being in the British Museum, but since he wrote the archive has been transferred to the British Library.
[20] For example (Becher, 1980) and (Pycior, 1981). However, (Fisch, 1994) has also looked at the original, but does not add much to the other essays about this.
[21] (Babbage, c.1821, p. 47)
[22] (Dubbey, 1978)

likely that these papers were written long before they were actually published[23], so it is impossible to establish when exactly Babbage stopped his mathematical work.

The work itself is divided into ten chapters, of which the third is relevant to algebra, entitled *'General Notions Respecting Analysis'*. This outlines his ideas on the development of algebra, and how it could be extended to wider concepts than just arithmetic. The first few paragraphs of the essay outline how Babbage saw the development of algebra through time: from merely the representation of an unknown number that was to be calculated through to a representation of a large variety of things. Indeed, the fourth paragraph provides a good summary of this:

> *'Thus did letters whose signification was at first restricted to pure number gradually acquired other secondary meanings and in various situations they denoted time, space, direction, and a variety of other circumstances.'*[24]

The program of his essay is set out here; he was trying to generalise the algebra that has been developed, particularly that which had not been established rigourously. It is important to note that Babbage was trying to extend algebra to cover more than the complex numbers that Woodhouse was trying to enlarge it to encompass in the first few years of the century, instead he is aiming to generalise it to a full variety of objects.

The major concept that Babbage has in his work is to develop each operation through a separate definition, much in the sense of Woodhouse, but he is much less concerned with the series definitions, particularly of exponentials, and instead uses definitions such as $x^a \times x^b = x^{a+b}$ as the fundamental definition of exponentiation for other quantities. Indeed, he says:

> *'When a and b are whole numbers it followed from the same premise that*
>
> $$x^a \times x^b = x^{a+b}$$
>
> *This equation, which is true for whole numbers does not necessarily subject when the exponents are fractions: still less does it necessarily subject when a and b are imaginary quantities.*
>
> *In order to give meaning to such expressions, we must have recourse to a new definition, and to avoid ambiguity it is extremely convenient that the new definition should include the old one as a particular case. This has been accomplished by assuming the equation*
>
> $$x^a \times x^b = x^{a+b}$$
>
> *is the definition of the operation denoted by the application of the exponents to the quantity x.'*[25]

There are a few other points that it is worth considering. Firstly, this definition is given in the most general sense possible: there are no restrictions at all placed on a and b, which shows how general

[23] Ibid. p. 166
[24] (Babbage, c.1821, p. 41)
[25] Ibid. p. 43-44

Babbage wanted his algebra to be. Secondly, the use of the words *'imaginary quantities'* is significant in that it is the only reference to complex numbers in the whole essay, giving an approach that differs significantly from that of Woodhouse. Thirdly, Babbage afterwards proposes that the other fundamental operations must have *'corresponding extensions'*[26], which demonstrates that Babbage was designing a comprehensive system that encompassed all of algebra.

Another important point concerning Babbage's work is he was the first person to attempt to provide a distinction between arithmetic equality and a form of algebraic identity, although the distinction is not nearly as profound as that made by Peacock. He did this when discussing infinite series, in that he ruled that infinite series were not legitimate in algebra, but that they are arithmetically valid. Instead, he preferred to allow only what he referred to as indefinite series, that is series which consist of a finite but general number of terms, and then use a remainder term[27]. Babbage then indicated that he would allow infinite series solely in the domain of arithmetic only when the series is convergent, in the sense that the remainder term decreases to zero. This is given in the following section, where consideration is given to the series for $\frac{1}{1-x}$, written in the indefinite form $\frac{1-x^{n+1}}{1-x}=1+x+x^2+x^3+...+x^n$:

> *'... when such a result occurs in any enquiry where a numerical result is the final object (as it most frequently is), we must make use of those rules which apply to the arithmetical interpretation of algebraic signs. The actual value of x must then be taken into consideration if n is supposed infinite, and then the formula will only be numerically accurate, or rather will afford an approximation only when x is some number less than unity.'*[28]

It is important to notice that Babbage was not explicit in what he says here, so it is only a hint that is taken from this, rather than any fully substantiated inference. The fact that these series were considered to be solely arithmetical and not to exist in the realm of algebra is important in that it shows a difference between an arithmetical and an algebraic equality. When it comes to discussing Peacock it will be seen that, although he was developing a similar notion, he interpreted it in a very different manner.

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[26] Ibid.

[27] Ibid. p. 47-48

[28] Ibid.. It is unclear exactly how Babbage is defining his convergence criterion here: it could be just that the terms are getting smaller, or that the remainder quantity itself is decreasing to zero, since both are supported somewhat by the text.

Peacock's Work on Algebra

George Peacock was the next major driving force in the development of British algebra, contributing three works in 1830, 1833 and 1842-45. The first and last of these were two editions of his '*Treatise on algebra*', whilst the 1833 work was in the form of a report to the British Association for the Advancement of Science. These three works all approach his ideas on algebra in slightly different ways, but they are all aiming for the same overarching concept. In particular, the second edition of his *Treatise* is much clearer in its exposition of the concepts than the earlier version.

The content of the three documents is also different, in that whilst the various editions of the *Treatise on Algebra* (the second of which is in two parts) were essentially meant to be textbooks written for students, whilst the *Report* was written for other mathematicians, so is helpful in understanding the mathematical content without the need to read large parts trying to explain basic algebraic rules.

It is worth at this stage noticing that Peacock was a member of the Analytical Society, so knew both Babbage and Bromhead well. Indeed, there is a letter from Peacock to Babbage explaining how he had read what appears to be Babbage's *Philosophy of Analysis*. A larger description of the differences and similarities between the works is given later.

The major concept of Peacock's algebra that is known about is his *'principle of the permanence of equivalent forms'*, which is discussed in the next few sections.[29] It was developed as his way of justifying the parts of algebra that were seen as new at the time. The idea that he had was to divide algebra up into two parts – arithmetical and symbolical algebra, and then to introduce the idea of the principle of the permanence of equivalent forms in order to enable results to be transferred from one to the other. Essentially, when a result in arithmetical algebra could be written in a general form, it could then be used in symbolical algebra by this principle.

Peacock's Aims

The basic aim of Peacock when he derived his system of algebra was to give all of algebra the status of a *'science'*, that is, to give it a justification so that it could be known to be true, and to therefore end any controversy over whether negative and complex numbers were valid for use, which had been raging in Britain for well over 50 years. By the time that he wrote his various works, almost everybody was comfortable with the uses of these numbers; no-one had managed to place them on a secure footing.

Peacock acknowledged the historical lack of consideration, saying, *'Algebra, considered with reference to its principles, has received very little attention, and consequently very little improvement, during the last century.'*[30] He also said that this *'regards its completeness as an*

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[29] Much of what follows in this chapter was originally a separate essay.
[30] (Peacock, 1834, p. 198)

independent science[31]', indicating his desire to put his branch of algebra on the most secure setting possible.

This is important, as it gives an understanding into what Peacock was trying to achieve with his new system of algebra – rather than aiming for being able to construct new results with it, he is only trying to justify what others have already managed to discover. In this context, it is the ideas behind what he says that are important, to probably a greater extent than the content itself.

Arithmetical Algebra

The system of algebra that Peacock devised involved separating algebra into two distinct parts[32] – arithmetical algebra and symbolical algebra. The former was designed to encompass only those things which at that stage were considered to be established without doubt, whilst the latter could encompass many of the new innovations that, although not previously justified in the 1830's, were being widely used in mathematics.

The distinction between arithmetical and symbolical algebra is an important concept in understanding the principle of the permanence of equivalent forms, and a brief discussion is given here as to the division between the two portions. In arithmetical algebra the principle restriction was that numbers could only be considered if they were non-negative, so that subtractions needed to be done the correct way round. For example, Peacock wrote, *'In the expression a-b, it is presumed that the number represented by a is greater than the number represented by b: if this condition is not satisfied then the operation of subtracting b from a could not be performed'*[33], and the result of this operation was then deemed to be *'impossible'* or *'imaginary'*.

There are a couple of areas where it is worth saying exactly where the boundaries of arithmetical algebra were drawn, as these form the area in which Peacock gave the examples by which he explained how arithmetical and symbolical differed, and then where he subsequently uses his principle of the permanence of equivalent forms. The first of these is in exponentials, with particular reference to the binomial expansions of series of the form $(1+x)^n$. Here, exponentials where the index is a natural number were fine; these were defined using the repeated product of the base. This could also be used to reach the binomial series where this series terminates[34].

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[31] Ibid.

[32] Peacock only divides the portion of algebra that is useful, and he explicitly allows for the concept of other algebraic structures. Indeed, Peacock comments that '*The process, therefore, by which we pass from one science to the other is not an ascent from particulars to generals, which is properly called generalization, but one which is essentially arbitrary, though restricted with a specific view to its operations and their results admitting of such interpretations as may make its applications most generally useful.'* (Peacock, 1834, p. 194) This demonstrates that other ways of modelling algebra would have been acceptable, but would not be useful, as without the correspondence with arithmetical algebra the results would not be able to be used in normal use. Indeed, there is evidence that Peacock applied this in that, prior to the publication of the second volume of the second edition of his *Treatise on Algebra*, he received a letter from Hamilton (Hamilton, Letter to George Peacock, 1844) that described the quaternions, stressing the way in which they breach commutativity. However without any applications, Peacock would not be concerned that this upset his system of algebra.

[33] (Peacock, 1842, p. 7)

[34] Ibid p. 23

Secondly, arithmetical algebra was defined only to contain those series which converged, and those that diverged were not taken to be arithmetical.[35]

Symbolical Algebra

Peacock proceeded to define symbolical algebra as an extension of arithmetical algebra where the operations and manoeuvres that were impossible in arithmetical algebra were then considered to be legitimate. In the introduction to the first volume of the Treatise on Algebra, Peacock writes that *'Symbolical algebra adopts the rules of arithmetical algebra, but removes the restrictions'*[36]. In removing these restrictions, Peacock did not just want to allow his algebra to work on negative, complex and other number systems, but he was trying to reach a subject where the important concept is algebraic relations, rather than their interpretation in any particular case.

The idea that Peacock had is that symbolical algebra should be an extension of arithmetical algebra, in that at each point at which an identity derived in symbolic algebra can be interpreted in arithmetical algebra, it should be true there. This statement has two important parts to it, which are that arithmetical algebra is embedded in symbolical algebra and that the process of interpretation was controlled. These two concepts are important for the development of the idea of the principle of the permanence of equivalent forms.

Peacock placed arithmetical algebra in symbolical algebra by adopting the same rules to govern the various operations, except for removing the various constraints that were placed on the arguments of the operations. One important thing to note is that there is no requirement for the operations to have the same meaning in symbolical algebra, only that they must obey the same set of rules. This allows Peacock to use his system of algebra for, say, the addition of geometrical lines without having to assume they are numbers[37], which makes his algebra more general.

The idea of arithmetical algebra being embedded in symbolical algebra was often expressed by Peacock using the term *'science of suggestion'*[38], and this is an important term for him to use. What he meant by this is that the rules of arithmetical algebra that have been deduced from the definitions there then will form the (new) definitions that he used in symbolical algebra. This is the key link that allows for the relationship between arithmetical and symbolical algebra to be exploited, which is what the principle of the permanence of equivalent forms then was able to do.

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[35] (Peacock, 1845, p. 27) Peacock here gives the definition of convergent and divergent series as follows: A convergent series is a series *'whose terms diminish and are ultimately zero are called convergent series: whilst those whose terms perpetually increase, and which exclusively belong to Symbolical Algebra, are called divergent series'*. (Peacock, 1845, p. 27). However, it is worth noting that Peacock is only considering series that come as a result of expanding algebraic expressions, and not just summing arbitrary lists of numbers, so the results he gets are all comparable to other formulae that he can evaluate, and whether the series converges to the value that the equivalent formula gives defines whether or not the series is arithmetical.

[36] (Peacock, 1842, p. vi)

[37] In this case what he is actually doing is treating two lines of the same inclination and length as the same, and this is equivalent dealing with complex numbers in the Argand diagram. However, the important point is that this relation does not have to be made for him to be able to use those objects in his algebraic system.

[38] (Peacock, 1834, p. 198)

The important thing to note here is that the ideas from arithmetical algebra were being treated as the definitions of symbolical algebra, as Peacock mentioned, for example he said, *'The rules of symbolical combination which are thus assumed have been* ***suggested only*** *by the corresponding rules in arithmetical algebra. They cannot be said to be founded upon them, for they are not* ***deducible*** *from them'*[39] (The emphases here are Peacock's own). It is worth noting here that Peacock was clearly setting out saying that he had not derived the laws in symbolical algebra, especially for when it comes to understanding the permanence of forms.

The way in which interpretation was controlled was through the distinction between arithmetical equality and algebraic equivalence. The former was the standard notion of equality, where substituting in any (arithmetically valid) number produced the same value on each side, whilst the latter denoted a more general concept; that one side could be transformed into the other by means of algebraic manoeuvres. Peacock gives an explanation to this in the corrections section of the second volume of his *Treatise*[40], where he shows how this designation works.

He gave an instructive example, where, considering the binomial expansion of $(1+x)^n$, that is $(1+x)^n = 1+nx+n(n-1)\frac{x^2}{1.2}+\ldots$, which was an algebraic equivalence – this became an arithmetic equality under the conditions that n is positive and rational (so that the exponent could be interpreted in arithmetical algebra) and that $0 \le x < 1$, so that the series converged.

To Peacock, this then meant that he could give a good definition of when an interpretation was legitimate, and this allowed him to give the idea of specialisation, which to him seemed to be reasonably obvious. He spells it out as follows: *'Whatever form is algebraically equivalent to another when expressed in general symbols must continue to be equivalent whatever those symbols denote.'*[41] It is important here to note that the forms were only *'equivalent'*, rather than being equal, as it is not always the case that an expression in symbolic algebra could be understood in arithmetical algebra.[42]

The above quotation is supported by the justification that Peacock gave, which makes clear why Peacock thought that the statement was obvious. He wrote that the above statement *'must be true, since the laws of combinations of symbols by which such equivalent forms are deduced have no reference to the specific values of the symbols'*[43]. To Peacock, this was an obvious statement to make, and the reasons that it does not quite seem to fit to a modern reader are discussed below.

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[39] Ibid.
[40] (Peacock, 1845, pp. 449-450)
[41] (Peacock, 1834, p. 198)
[42] In earlier versions of his work, Peacock considered this to be a part of the principle of the permanence of equivalent forms. The ideas that he expresses there are the same; it is solely a matter of nomenclature, but one that it is worth being aware of.
[43] Ibid.

The Principle of the Permanence of Equivalent Forms

The principle of the permanence of equivalent forms was the way of using results from arithmetical algebra in symbolical algebra and vice versa. The statement of the principle, as Peacock wrote it in the second edition of his Treatise on Algebra (1845) was:

'Whatever algebraic forms are equivalent, when the symbols are general in form but specific in value will be equivalent likewise when the symbols are general in value as well as in form.'[44]

In terms of the language used earlier, this statement is really saying that whenever an arithmetical equality is in such a form that it can be read as an algebraic equivalence then it is one. There are nuances within this though, as what constitutes being in such a form was not necessarily obvious. In essence, therefore, it needed to be written in such a way that there is no dependence upon any of the terms being an integer, or being positive real numbers.

For example, when dealing with indices, for all positive integers m and n, it is possible to say both $\underbrace{aa...a}_{n} = a^n$ and $a^m a^n = a^{m+n}$, where the derivation of the second identity comes from splitting it up using the first definition, and then using the definition again on the new product. However, of these, only the latter was applicable for use with the principle of the permanence of equivalent forms – it does not in its form use the fact that m and n are integers (although it does in the derivation).

The idea behind this was the concept that arithmetical and symbolical algebra were required to coincide exactly whenever they considered a common result. Peacock expressed this, saying things like, *'the absolute identity of the results of the two sciences as far as they exist in common'*. This was really equivalent to the principle of the permanence of equivalent forms combined with the idea of interpretation that is discussed above.

It is worth at this stage noting that Peacock used the word *'operations'* to refer to two separate concepts. He used it both in the modern sense, to refer to things like addition and multiplication[45], and to refer to *'every process by which we pass from one equivalent form to another'*[46]. In the discussions here, the word operation is retained solely for the former, whilst the word 'manoeuvre' is preferred for the latter, in order to avoid ambiguity.

To clarify exactly what Peacock meant when he discussed his principle of the permanence of equivalent forms, below is given an example of when it can be used in order to derive a result that would otherwise not be obtained in arithmetical algebra.

The binomial theorem can be written in the requisite form, and this is a case used by Peacock on in his *Treatise on Algebra*[47]. Written as $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{1.2} + ...$, it is, as above, in the correct form, and can thus be taken to be valid for general n, not just those that are integers by this

[44] (Peacock, 1845, p. 59)
[45] (Peacock, 1842, p. 52)
[46] (Peacock, 1845, p. 60)
[47] Ibid. p. 110

principle. For example, in symbolical algebra, it was fine to write $(1+x)^{\frac{1}{2}} = 1+\frac{x}{2}-\frac{x^2}{8}+\ldots$, whereas the fractional index would not be considered valid in arithmetical algebra.

Comparison with Babbage's *Philosophy of Analysis*

In his two works on Babbage, Dubbey[48] proposed that the major content of Peacock's work was foreshadowed in Babbage's *Philosophy of Analysis*. This hypothesis does not seem to give Peacock his fair due for the changes that he made to Babbage's hypotheses. Both Babbage and Peacock in their works were attempting to justify algebra in a rigourous sense, moving beyond arithmetic into a general system. They also both considered indices as a principal example.

These similarities are in some ways to be expected; it is known that Peacock read Babbage's work because there is a letter from Peacock to Babbage referenced by Dubbey[49] in which Peacock discusses reading Babbage's work.

However, there are many ways in which the works have different perspectives, as would be normally expected given that Peacock was aiming to improve on the algebras that had gone before, not just to restate them. Peacock seemed to react much more fully to the work of Woodhouse, in that he discussed complex numbers as part of his algebra much more than Babbage does[50]. Peacock was also a lot stricter in what he allowed in from arithmetic, forbidding all forms of negative numbers, whereas Babbage has no qualms with using these without any new principles.

The most major difference between the two systems is the way in which their authors constructed their generalisations of algebra. Whereas Babbage aimed to provide a new definition for each operation in order to extend them, Peacock develops his one main theory which encompasses all the operations in one go. This is a huge alteration that Peacock makes, and it is important to draw the distinction, even though, from a modern perspective, Babbage's approach seems to be closer to that which is used today.

Finally, the consideration of infinite series is entirely complementary. Both Babbage and Peacock only admit convergent series (subject to the definition of convergence), but whereas Babbage removes entirely infinite series from his algebra, Peacock's symbolical algebra includes all series that come from an algebraic expansion, even when they did not necessarily converge.

[48] (Dubbey, 1977) and (Dubbey, 1978)
[49] Ibid.
[50] He discusses their geometry also, in the tradition of Buée and Warren (see the next section).

Complex Numbers and Geometry

The story of Complex numbers in geometry is one that is, as stated before, not as important as that of algebra, but nevertheless worth commenting on, as it is an important part of the development of algebra in the period.

Buée

The first person to discuss the geometry of complex numbers in Britain was Adrien Quentin Buée, a French émigré from the revolution who was living in London[51]. The major paper that he wrote on this was entitled *Memoires sur les Quantitiés Imaginaires*, and was published in the Philosophical Transactions of the Royal Society in 1806, in French[52].

The major hypothesis of Buée's work is that √-1 is solely a geometrical concept – and it denoted perpendicularity. Indeed, he said that:

> *'√-1 n'est donc pas le signe d'une opération arithmétique, ni d'une operation arithmético- géométrique (No. 2), mais d'une operation purement géométrique. C'est une signe de perpendicularité.'*[53]

It is important to notice that Buée explicitly ruled out any form of hybrid interpretation with his combined term '*arithmético-géométrique*' – he was considering solely the geometrical interpretation of √-1.

One major point that Buée made is to distinguish between two types of 0 (and thus implicitly between two types of number).

> *'Le signe 0 a deux significations. On peut en effet le considérer sous un point de vue arithmetique et sous un point de vue descriptif. Sous le premier, 0 signifé quantité nulle. Sous le second, il signifé une description telle que la distance entre le premier et le dernier point est nulle.'*[54]

This distinction is an expansion of the division made earlier into arithmetic and geometric spheres, making it clear that √-1 cannot exist in arithmetic, but can exist in geometry, which is viewed as a separate science.

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[51] (Didot, 1853, p. 730)

[52] (Buée, 1806). At this point I should confess that my French is not of a sufficient quality to assume that the translations are perfectly accurate.

[53] Ibid. p.29. This translates as, *'$\sqrt{-1}$ is therefore not the sign of an arithmetic operation, nor of an arithmetic-geometric operation, but of a purely geometric operation. It is a sign of perpendicularity.'*

[54] Ibid. p. 36. This translates as, *'The sign 0 has two meanings. One can, in effect, consider it with an arithmetic point of view, and with a descriptive point of view. In the first, 0 means nil. In the second, it means such a description that the distance between the first and last point is zero.'*

Playfair's Response to Buée

A criticism of Buée's work was published in 1808 by John Playfair in the Edinburgh Review[55]. This appears to be the only immediate response to Buée's work; coming from a member of the eighteenth century school of algebra it is largely critical of the approach Buée takes.

His major objection was that √-1 could not represent anything actual, being an 'imaginary' or 'impossible' quantity. He did accept that parts of the work were useful, and concluded that more research was necessary. The key point remains that he rejected the central premise of Buée's work.

This whole approach is evidenced by the following quotation, which demonstrates the hardness of the opposition to the work:

> *'… there have been more than one attempt to treat imaginary quantities as things really existing, or as certain geometrical magnitudes which it is possible to assign.*
>
> *The paper before us is one of these attempts; and the author, though an ingenious man, and, as we readily acknowledge, a skilful mathematician, has been betrayed into this inconsistency by a kind of metaphysical reasoning, which we confess ourselves not always able to understand.'*[56]

This also indicates how the opposition that was partly due to an inability to grasp exactly what Buée was trying to achieve. It is true that the work is not the most elegantly phrased, and it is not totally explained, but nevertheless it does hold some weight as a theory.

Warren's '*On the Geometrical Interpretation*'

John Warren published his work '*A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*' in 1828[57], the first work on the geometry of complex numbers published in Britain that influenced later mathematicians[58]. The work is the first to explicitly note that '*adding 360°*', meaning, in modern terms, multiplication by $e^{2\pi i}$, does not change a quantity, but can change the result under certain operations.

The notation that he introduced was to write below the number how many extra complete revolutions it had, so $\underset{0}{1}$ and $\underset{1}{1}$ were both equal expressions for 1, but $\left(\underset{0}{1}\right)^{\frac{1}{2}} = 1$, whilst $\left(\underset{1}{1}\right)^{\frac{1}{2}} = -1$, since the extra revolution was halved to produce the minus sign. This notation was somewhat unwieldy to work with, but produced the results that give all the complex options that were needed in an obvious way.

Warren completely accepted the geometrisation as we would today have it in the complex plane and the full generalisation of this was given, allowing from the early definitions the addition of complex numbers.

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[55] (Playfair, 1808). Playfair spells Buée with an umlaut rather than an acute accent.
[56] Ibid. p. 306
[57] (Warren, 1828)
[58] Peacock certainly read Warren; see the preface to his *Treatise on Algebra:* (Peacock, 1830, p. xxvii).

It is important that Peacock’s Treatise on Algebra was published in 1830, only two years after Warren’s book, so that Peacock was well into his writing, and he acknowledged as such in the preface. They were both discussing a similar concept, and there is more to say about this in the next section. Warren’s work is well regarded by De Morgan who said that:

> *‘Mr Warren was the first who combined the various hints which had been given that ‘√-1 denotes perpendicularity’ into a connected, demonstrated logical system.’*[59]

This shows that the Warren, independently of Peacock, had a system of geometrical complex numbers that were well regarded and useful, in the eyes of his contemporaries.

Peacock’s Work on Complex Numbers and Geometry

The contribution of Peacock to the development of complex numbers in geometry is less than he made in the theory of algebra, mostly because some of what could have been new was instead contained within Warren’s book, as well as the fact that the major thrust of his work was towards a new theory of algebra, and the application of this to complex geometry was a lesser point. Unlike with the work on algebra, this is only developed really within the two editions of his *Treatise on Algebra*, and not within the *Report*.

Peacock’s first development into the geometry of complex quantities was to extend the concept that he had of *‘signs of affection’* to allow for more signs than just ±1, in particular to allow for complex roots of both 1 and -1. He combined this with the idea of perpendicularity being denoted by $\sqrt{-1}$ that is exhibited in both Buée and Warren, and from this generated the concept of other arguments in the complex plane.

Peacock also explores the uses of de Moivre’s theorem in the complex plane. It is important to note that de Moivre’s theorem had been known to mathematicians in Britain for a considerable time, for example Woodhouse mentions it in his 1801 work[60] , although he does not, as Peacock does, refer to de Moivre by name. Peacock also extended this to deal with other trigonometric identities, which was useful, but not interesting from a historical point of view.

De Morgan

Augustus De Morgan was the next person to develop the concept of complex numbers themselves further, but this is beyond the scope of this work.

[59] (De Morgan, Review of A Treatise on Algebra by George Peacock, 1835, pp. 303-304)
[60] (Woodhouse, 1801, p. 98)

Criticism of Peacock

There are three major criticisms of Peacock's work that were published prior to De Morgan's major works on algebra, principally his 'On the Foundations of Algebra', published in four parts between 1839 and 1844. These were written by William Rowan Hamilton, Osborne Reynolds and Philip Kelland[61], the last of which is the publication of some lectures given by the author.

These are the only commentaries that were written in the aftermath of Peacock's work, and give a good indication of the response to Peacock that was experienced within Britain, although the scarcity of the criticism is perhaps also a good indicator that, for a time, Peacock's work was not particularly challenged by many within Britain.

There are also a pair of works published on the pedagogical value of Peacock's work, Augustus De Morgan's *Review of Peacock's Treatise on Algebra*[62] and William Whewell's *Thoughts on the study of mathematics as part of a liberal education*[63] which are respectively in favour and critical of the work, but only in its value as an educational tool, not as a work on the foundations of algebra.

William Rowan Hamilton

William Rowan Hamilton response to Peacock was not overt; he is not mentioned by name in Hamilton's 1837 work, and the reference is known through a letter that Hamilton had sent to Peacock in 1835, where he explained to Peacock the reference:

> *'The introductory remarks, however, without expressly mentioning you, recognise the existence (I might say the necessity) of a Philological School of Algebra, in which school you perhaps have been the most bold and consistent teacher.'*[64]

Thus, the references to the *'Philological'* style of algebra in his essay can be taken to be references to Peacock's work.

The criticism that Hamilton levelled at Peacock was, in essence, that the method that Peacock had been using to consider algebra was too concerned with how to develop the language, rather than with actually developing algebra as a science. This is, from a modern perspective, a valid criticism in that Peacock has not actually justified fully his principle of the permanence of equivalent forms, on which so much of his idea relies. Hamilton talked of removing the *'confusions of thought, and obscurities or errors of reasoning'*, and not the '*failures of symmetry in expression'*[65]. It is, however, important to note that Hamilton was a more isolated voice among the group that were considering algebra, and it does not necessarily follow from the fact that he was critical that in fact everyone was critical.

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[61] (Hamilton, 1837), (Reynolds, 1837) and (Kelland, 1843) respectively.
[62] (De Morgan, 1835)
[63] (Whewell, 1835)
[64] (Hamilton, 1835)
[65] (Hamilton, 1837, p. 294)

Reynolds' Strictures

The second criticism to be considered is the 1838 work of Oswald Reynolds, who had recently graduated from Queens' College[66], which was entitled *Strictures on Certain Parts of Peacock's Algebra by a Graduate*[67]. It was published anonymously, and the sources attributing it seem to be rather limited. De Morgan cited it as being Reynolds in 1849[68], and it appears that he had this information in a letter from Sylvester at some point between 1838 and 1841[69].

The most important source on this is Pycior's paper *Early Criticism of Symbolical Algebra*[70], which is seemingly the only place in the literature that considers Reynolds' work in any detail.

The work itself is split into five areas of criticism, which are the '*Principle of the Permanence of Equivalent Forms*', '*Symbolical Algebra*', '*On the Independent Use of the Sign –*', '*On the real nature of* number' and '*On the correct representation of algebraic* symbols'. Each of these is a criticism of a different part of the system of algebra that Peacock constructed.

The contention in the first of these sections was that the meaning of generality is too loosely defined, allowing the admittance of things which do not agree. For example, he complained about specialisation where some of the terms in a sum become zero, or in a product becomes one, in that this is not, in fact the same form. He also brought up the form of sums that are given to n terms, such as the finite expansion of a binomial form, which does not extend to a general product, although Peacock specifically excluded such a case.

The criticisms that Reynolds gave in the second part of the work seem to be more worthwhile and less just misunderstandings of what Peacock was trying to say. Firstly, Reynolds opposed the way in which the symbols in symbolical algebra seemed to simultaneously have two distinct meanings: they were both formal symbols that had no meaning except for being themselves as well as able to stand for anything in total generality.

However, in this portion, there is also an objection to the fact that some of the symbols could not always be interpreted in a given context. An example that is given here is the meaning of $\sqrt{-1}$ units of time, which was, to Reynolds, clearly absurd. The same applies to an imaginary value of an angle, but Peacock appears to have answered it when he explained that the application relied on the symbols being able to be comprehended in the context in which they were used.

Another use of the same idea is the objection that Reynolds had to the extension of operations in ways that were not in line with their initial definition, for example allowing multiplication by non-integer values as that does not correspond with the definition of repeated addition. This seems to miss the reason why Peacock needed to introduce his principle: it was precisely because the definition could not be naturally extended in such a way that the principle was necessary.

[66] (Venn & Venn, 1940-54)
[67] (Reynolds, 1837)
[68] (De Morgan, 1849)
[69] This is given in (Pycior, 1982).
[70] Ibid.

Both of these first two criticisms can be taken as being essentially misconceptions of what Peacock's system aimed to do. It was aiming to be the justification, not to be immediately justifiable from the previous systems of algebra, since, if it were, then there would have been no need for Peacock to spend so long examining it.

The third section is a question of what exactly is meant by the use of the signs *'+'* and *'-'*. It suggests that the signs themselves were not properly defined, since they were supposed to not just represent an operation, but exist independently of it. This was a fair criticism, and Peacock was fairly unclear exactly what he means by the sign '-', although '+' was better able to be comprehended as not changing the magnitude. The exact meaning of '-' to Peacock would be something that depended on the situation, as long as it made sense in some way that was consistent with how the minus sign was supposed to operate.

The last two stages are not criticisms per se of Peacock's work, but are instead small expositions of the way in which Reynolds believes that algebra should be treated. He first concludes that number is a different form of quantity from anything else, since it is entirely an abstract concept, saying that:

> *'From the preceding remarks we conclude that number differs from all other quantities in this respect, viz. that it is perfectly abstract. Nothing else exists which does not contain specific properties that distinguish it from other things. Moreover there is no other quantity from which number may not be abstracted; therefore, first, number is perfectly abstract; secondly, it is the only thing which is so.'*[71]

The idea here is that numbers are treated differently from the other quantities that are encountered in algebra, in contrast to Peacock's concept where the same algebra applies to all of the types of quantity and treated them the same.

The fifth and final section continues to outline how the construct of negative numbers is developed, based on the general relations which quantities $+a$ and $-a$ must satisfy with respect to each other, rather than in absolute terms. It then goes on to claim that algebra could only consider numbers, but that, as numbers can be used to represent a large variety of other things, problems involving things that are not pure number can also be solved.

Overall, Reynolds presented a criticism of Peacock that appears to be partially, but not totally valid, and then presented his own version of algebra that is interesting, but does not appear to be historically significant, in that he was not well known enough for the leading mathematicians who would go on to develop algebra over the following period to necessarily pay his work much attention: indeed he is not referred to in the prominent works of the next few years.

[71] (Reynolds, 1837, p. 21)

Kelland's Lectures

Philip Kelland published in 1843 a book entitled *Lectures on the Principles of Demonstrative Mathematics*[72], which was in fact a write up of the lectures that he had been giving for some time, both in Cambridge as a fellow of Queens' college, and then in Edinburgh as a professor of mathematics there[73]. It is interesting to notice that Kelland in his lecture notes referred to the latest developments in algebra, whilst a set of lecture notes that are purportedly Peacock's from a very similar time do not.[74]

The complaint that Kelland had with Peacock's symbolical algebra is that he felt that it went the wrong way between arithmetic and algebra, and should start with arithmetic and then extend it, rather than allowing it to be found as a part of the larger algebra. In this, he was proposing a theory along a similar line to Babbage; although there is no way in which he could have read Babbage's work, only coming up to Cambridge in 1830[75].

Kelland then pointed out that one difference between the two systems is that *'we run no risk of making inconsistent assumptions of things that depend on each other'*[76], since the definitions were merely extended, and there is none of the two way manoeuvring that Peacock uses in his system. Furthermore, he was quite explicit in that it was a problem with this two way conversion between arithmetical algebra and symbolical algebra that concerned him, saying that *'we are under the necessity of applying the principle of the permanence of equivalent forms in both its features at the same time'*[77]. It should be noted here that this is the 1830 use of the name, not the version in use in the second edition of Peacock's *Treatise on Algebra*.

He also opposed it for what were essentially pedagogical reasons, asking *'whether the mind is more readily capable'*[78] of using one system of algebra rather than the other, but this is not so important in the scheme of the development of algebra.

[72] (Kelland, 1843)
[73] (Venn & Venn, 1940-54). Pycior, in (Pycior, 1982, p. 410), notes that both Kelland and Reynolds were at Queens', hypothesising that the college may have been a centre of opposition to Peacock's view on algebra, but there seems to be no other evidence to support it.
[74] (Peacock, 1836). The reference that these are Peacock's is solely due to De Morgan in (De Morgan, 1849), the same work in which he gives the reference for Reynolds. There seems to be no reason to doubt this, as De Morgan would have known the various people who were involved. However, it is worth knowing that in the Cambridge University Library, the work is listed as having been written by Henry Pearson.
[75] (Venn & Venn, 1940-54)
[76] (Kelland, 1843, p. 121)
[77] Ibid. p. 122
[78] Ibid. p. 123

Appendix

Biographical Information

What is included here is not a full biography of any of the people mentioned, but only a brief summary of what is known of the lives of those people who feature particularly prominently, or are otherwise not well written about. People such as Charles Babbage, William Rowan Hamilton and Augustus De Morgan are well studied and documented, so they are omitted from this list.

Buée, Adrien Quentin. Born in Paris in 1748, he also died there in 1826. He entered the priesthood, and was seemingly at some point an abbot, as he is often referred to with the title *'M. l'Abbé'*. He stayed in Britain for 21 years over the period of the French revolution; it is not unsurprising that he did not support the revolution, being a cleric.[79]

Kelland, Philip. Born in Lancras, Devon in 1808, he matriculated at Queens' college in 1830, and graduated as senior wrangler in 1834. He was appointed a fellow there in 1835, and moved to become the first English professor of mathematics in Edinburgh, a position he held until his death. He was a well known educational reformer in Scotland, and died in 1879.[80]

Peacock, George. Born 9th April 1791, he was admitted to Trinity College Cambridge in 1809, where he was a founder member of the Analytical Society. He was the second wrangler in 1812. Appointed a fellow in 1814, and then a tutor in 1824, a position which he held until 1839, he became the Lowndean professor of Astronomy and Geometry in 1837. He appears to have done little mathematics after his appointment in 1839 as Dean of Ely, concerning himself with university administration (he was at one stage secretary to the chancellor) and ecclesiastical matters. He died in London on 8th November 1858.

Reynolds, Osborne. He matriculated at Trinity in 1833, and migrated to Queens' the following January, graduating as the 13th[81] wrangler in 1837. He was long term a schoolmaster in various places across Britain, including Belfast, Buckinghamshire and Essex. He died in Northamptonshire in 1890. His son, Osborne Reynolds, is the man of Reynolds number fame in fluid mechanics.[82]

Warren, John. Born 1796, he was admitted to Jesus College Cambridge in 1814. He became a fellow there after his graduation in 1818; he later seems to have become a priest. He died in 1852. [83]

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[79] (Didot, 1853, p. 730)
[80] (Venn & Venn, 1940-54)
[81] (Venn & Venn, 1940-54) lists him as 13th wrangler, whilst (Pycior, 1982, p. 402) has him as the 12th wrangler.
[82] (Venn & Venn, 1940-54)
[83] (McConnell, 2004)

Further concepts to be studied

Martin Ohm is mentioned a couple of times as having something interesting to say, probably in something around 1826. However, this is normally in a reference to a group of other authors, and all the relevant works seem to only exist in German.

There is a large story here about the teaching of algebra and complex numbers and how that developed. In particular, there are numerous editions of James Wood's *Algebra* from the period, the already mentioned lecture notes of Peacock and Kelland as well as the criticisms of De Morgan and Whewell. The Tripos Questions of the period are furthermore published in two volumes:

Mathematical problems and examples, arranged according to subjects from the Senate-House examination papers, etc, Cambridge, 1837 (This covers 1821-35)
Cambridge Problems: Being a collection of the printed question proposed to the candidates for the degree of Bachelor of Arts at the General Examinations from 1801 to 1820 inclusive. Cambridge: Deighton.

Another fact of Interest

Peacock wrote in 1829 a work for the *Encyclopaedia Metropolitana*, entitled *A History of Arithmetic*, in which he attempted to cover the history of various numerical systems and how they developed. The bibliographical reference for this is:

Peacock, G. (1829). A History of Arithmetic, *Encyclopaedia Metropolitana vol. 1*, London: Smedley and Rose, p. 369-523.

There seems to only be one paper on this, that is:

Durand-Richard, M-J (2010), *Peacock's "History of Arithmetic", an attempt to reconcile empiricism to universality,* Indian Journal for the History of Science

It is worth noticing that this paper states that the encyclopaedia was first published in 1845, and, although the later volumes were, this first volume is clearly dated 1829 in the copy held in the Cambridge University Library.

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