University of Cambridge
Faculty of Mathematics
Post-Masters Consultancies

Shape Selection in Nematic Solids

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I. Research at the CSMLab

Research at the Computational Structural Mechanics Lab (CSMLab) is concerned with the computational modelling and analysis of light-weight structures relevant to applications in structural engineering, aerospace and marine engineering. Specific strengths include:

- advanced discretization methods for computational analysis of structural components such as membranes and shell
- modelling and analysis of large-scale structural membranes, such as parachutes and inflatable decelerators for outer-space applications
- computational analysis and simulation of fluid-structure interaction with applications to micro-air-vehicle design and insect flight

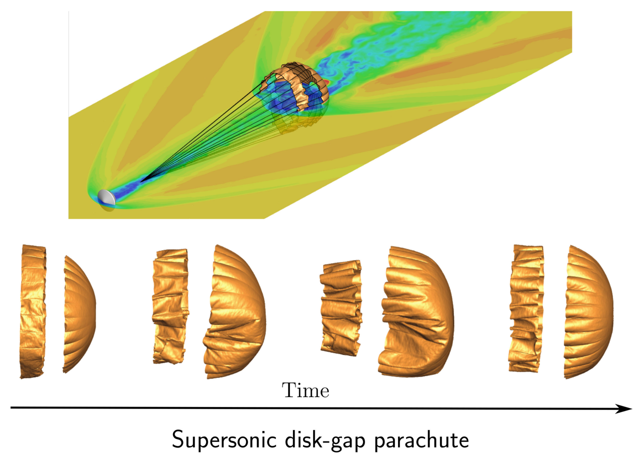

Figure 1 : Computational simulation of supersonic disk-gap-band parachute

Figure 2 : Photo Credit: Animal Flight Group, Dept. of Zoology, Oxford University

It is often the case that problems of interest in structural mechanics are associated with inherently complex dynamics and geometry. Such problems invariably demand advanced scientific computing and numerical analysis. For example, attempts to develop models of fluid-structure interaction in insect flight are complicated by the presence of the large deformations exhibited by biological wings during flapping motion, as well as the complex geometry of biological surfaces. Models that incorporate such complexities are needed in order to produce high-fidelity predictions for aerodynamic and structural design parameters. Specific computational tools used in this context include stabilized and immersed finite element methods.

II. The Finite Element Method

The finite element method (FEM) is a numerical analysis technique that is used to find approximate solutions to boundary-value problems. It provides the dominant discretization approach in structural mechanics and is used in the numerical modelling of physical systems in a wide range of engineering problems. The method is based on the variational formulation of the mathematical model and involves the subdivision of the geometry into disjoint components of simple geometry called finite elements. The response of each element is expressed as a finite number of degrees of freedom at a set of nodal points.

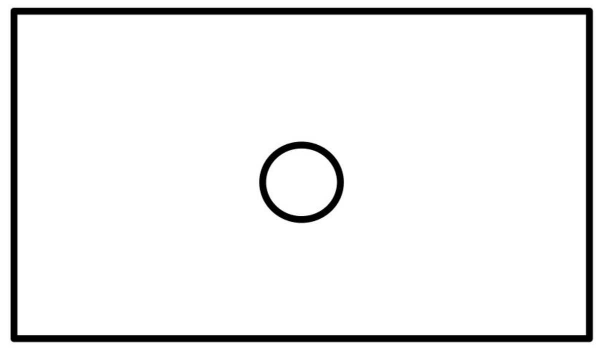

Figure 3 : Geometry of interest

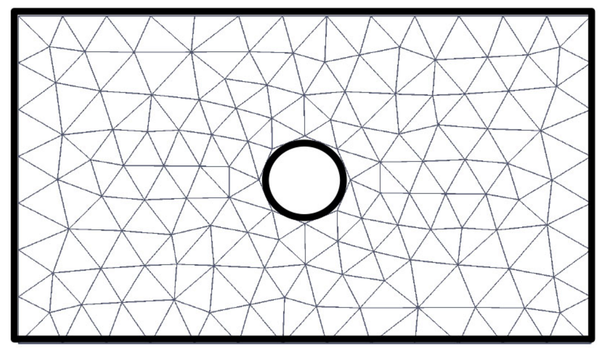

Figure 4 : Mesh-generation

Recall that the weak formulation of the equation $Au = f$, where V is a Banach space with dual space V', $u \in V$, $A : V \to V'$, $f \in V'$ is

$$B(u, v) = f(v) \qquad \forall v \in V,$$

where B is the bilinear form $B(u, v) = A(u)(v)$. The key steps involved in the FEM are:

- derivation of the weak formulation of the governing equations.
- mesh-generation
- interpolation of displacements and the test function on each element using shape functions and corresponding nodal values
- On each element, displacements u and the test function v are interpolated using shape functions N^K and nodal values u^K

$$u = \sum_{K=1}^{NP} N^K u^K \qquad v = \sum_{K=1}^{NP} N^K v^K,$$

  where NP is the number of nodes per element.
- The finite element equations are obtained by introducing these interpolation equations into the weak formulation. The solution is then found using numerical linear algebra.

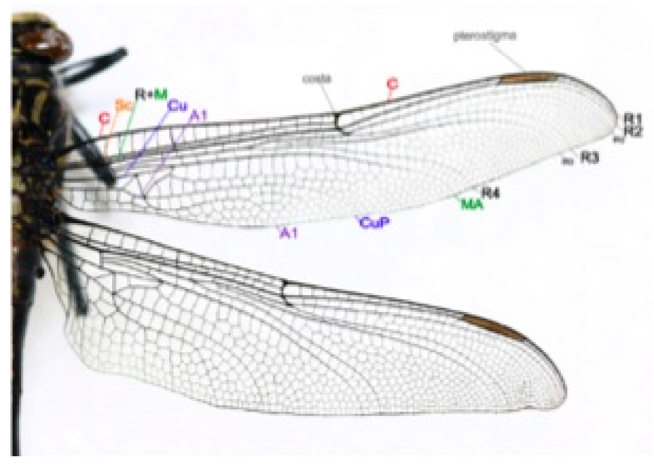

Figure 5 : Insect wing

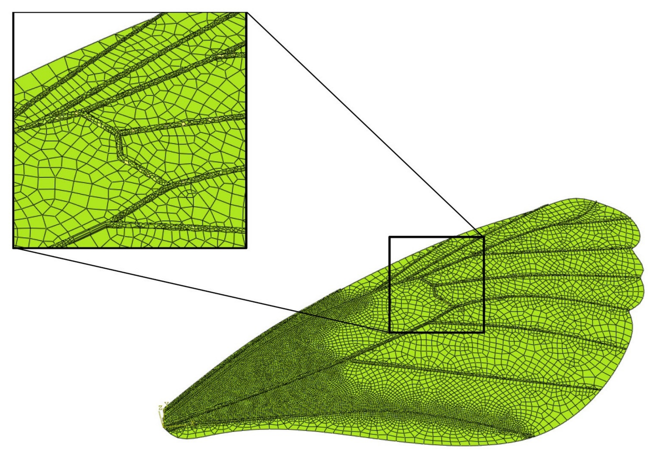

Figure 6 : FE mesh of wing

III. Beams, Shells and Membranes

Consider a beam of thickness t with domain

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R} : x_1 \in [0, L], x_2 \in [-b/2, b/2], x_3 \in [-t/2, /t/2]\},$$

whose deflections take place in the x_1x_3 plane, so that the midline is initially given by $x_3 = 0$. The Euler-Bernoulli beam model takes as its key kinematic assumption that the material points on the normal to the midline remain on the normal during the deformation. This assumption determines the axial displacement of the material points across the thickness. The Timoshenko beam model refines the Euler-Bernoulli kinematic assumpton to allow for the possibility of shear deformations. The Kirchhoff-Love theory of plates and shells is an extension of Euler-Bernoulli theory to these structures and is based on the assumption that material fibres normal to the mid-surface remain normal under deformation.

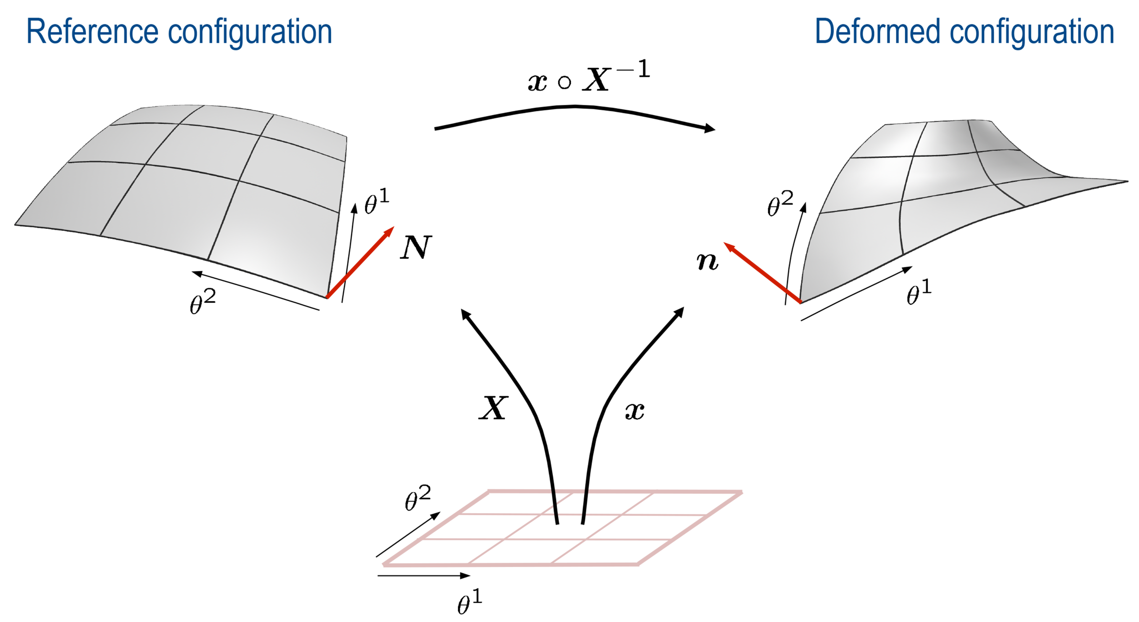

Figure 7 : Deformation of shell mid-surface in Kirchoff-Love theory

- Reference configuration: $\Phi = X + \theta^3 N \qquad \theta^3 \in [-t/2, t/2]$
- Deformed configuration: $\phi = X + \theta^3 n \qquad \theta^3 \in [-t/2, t/2]$
- Deformation gradient: $F = (\partial\phi/\partial\Phi)_{ij}$
- Green-Lagrange Strain:

$$E = \frac{1}{2}(F^T F - I) = (\epsilon_{\alpha\beta} + \theta^3 \kappa_{\alpha\beta})\partial\theta^\alpha/\partial\Phi \otimes \partial\theta^\beta/\partial\Phi, \quad \alpha, \beta = 1, 2,$$

where the membrane strain ϵ and bending strain κ are given by

$$\epsilon_{\alpha\beta} = \frac{1}{2}(\partial x/\partial\theta^\alpha \cdot \partial x/\partial\theta^\beta - \partial X/\partial\theta^\alpha \cdot \partial X/\partial\theta^\beta)$$

$$\kappa_{\alpha\beta} = -\frac{\partial^2 x}{\partial\theta^\alpha\partial\theta^\beta} \cdot n + \frac{\partial^2 X}{\partial\theta^\alpha\partial\theta^\beta} \cdot N.$$

The membrane strain measures the change in surface metric, whereas the bending strain provides a measure for the change in surface curvature.